Term Structure of Uncertainty in the Macroeconomy[*]

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Abstract

Dynamic economic models make predictions about impulse responses that characterize how macroeconomic processes respond to alternative shocks over different horizons. From the perspective of asset pricing, impulse responses quantify the exposure of macroeconomic processes and other cash flows to macroeconomic shocks. Financial markets provide compensations to investors who are exposed to these shocks. Adopting an asset pricing vantage point, we describe and apply methods for computing exposures to macroeconomic shocks and the implied compensations represented as elasticities over alternative payoff horizons. The outcome is a term structure of macroeconomic uncertainty.

Contents

1 Introduction

Impulse response functions quantify the impact of alternative economic shocks on future economic outcomes. In so doing, they provide a way to assess the importance of alternative sources of fluctuations. Building on the insights of Yule (1927) and Slutsky (1927), Frisch featured an important line of research on the "impulse and propagation problem" aimed at answering the question asking what are the sources of fluctuations and how they are propagated over time. An impulse, captured formally by the realization of a random shock, has an impact on an economic time series in all of the subsequent time periods. Response functions depict the intertemporal responses. Sims (1980) showed how to apply this approach in a tractable way to multivariate time series with a vector of underlying shocks, and he exposed the underlying challenges for identification. Subsequent research developed nonlinear counterparts to impulse response functions.

Macroeconomic shocks also play an important role in asset pricing. By their very nature, macroeconomic shocks cannot be diversified and investors exposed to those shocks require compensations. The resulting market-based remunerations differ depending how cash flows are exposed to the alternative macroeconomic shocks. We call the compensations risk prices, and there is a term structure that characterizes these prices as a function of the investment horizon. In this chapter we study methods for depicting this term structure and illustrate its use by comparing pricing implications across models. This leads us to formalize the exposure and pricing counterpart to impulse response functions familiar to macroeconomists. We call these objects shock-exposure and shock-price elasticities. Our calculations require either an empirical-based or model-based stochastic discount factor process along with a representation of how alternative cash flows with macroeconomic components respond to shocks.

There is an alternative way to motivate the calculations that we perform. A common characterization of risk aversion looks at local certainty equivalent calculations for small variance changes in consumption. We deviate in two ways. First, when making small changes, we do not use certainty as our benchmark but rather the equilibrium consumption from the stochastic general equilibrium model. This leads us to make more refined adjustments in the exposure to uncertainty. Second, movements in consumption at future dates could be induced by any of the macroeconomic shocks with occurrences at dates between tomorrow and this future date. Thus, similar to Hansen et al. (1999) and Alvarez and Jermann (2004), we have a differential measure depending on the specific shock and the dates of the impacts.

Empirical finance often focuses on the measurement of risk premia on alternative financial assets. In our framework, these risk premia reflect the exposure to uncertainty and the compensation for that exposure. Risk premia change when exposures change, when the prices of those exposures change or both. We use explicit economic models to help us quantify these two channels by which risk premia are determined, but a more empirically-based approach could also be applied provided that the uncertainty prices for shocks could be inferred. While there are interesting challenges in identification to explore, we will abstract from those challenges in this chapter.

Our chapter:

- defines and constructs a term structure of shock-exposure and shock-price elasticities applicable to nonlinear Markov models;

- compares these constructions to impulse response functions commonly used in macroeconomics;

- describes computational approaches pertinent for discrete-time and continuous-time models;

- applies the methods to continuous-time macroeconomic models with financing frictions proposed by He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014).

2 Mathematical framework

We introduce a framework designed to encompass a large class of macroeconomic and asset pricing general equilibrium models. There is an underlying stationary Markov model that is used to capture the stochastic growth of a vector of time series of economic variables. The Markov model emerges as the "reduced form" of a solution to a dynamic stochastic equilibrium model of the macroeconomy. Modeling stationary growth rates allows for inclusion of shocks that have permanent effects and nontrivial long-horizon implications for risk compensations. We provide a range of illustrative applications of this framework throughout the chapter, and we devote Sections 7 to a more extensive exploration of nonlinear continuous-time models with financial constraints.

We start with a probability space $(\Omega, \mathcal{F}, P)$. On this probability space, there is an n-dimensional, stationary and ergodic Markov process $X = \{X_t : t \in \mathbb{N}\}$ and a k-dimensional process W of independent and identically distributed shocks. Unless otherwise specified, we assume that each W_t is a multivariate standard normal random variable. We will have more to say about discrete states and shocks that are not normally distributed in Section 3.5.

The Markov process is initialized at X_0. Denote $\mathfrak{F} = \{\mathcal{F}_t : t \in \mathbb{N}\}$ the completed filtration generated by the histories of W and X_0. We suppose that X is a solution to a law of motion

$$
\begin{aligned}
X_{t+1} &= \psi\left(X_t, W_{t+1}\right) \\
Y_{t+1} - Y_t &= \phi\left(X_t, W_{t+1}\right)
\end{aligned}
\tag{1}
$$

The state vector X_t contains both exogenously specified states and endogenous ones. We presume full information in the sense that the shock W_{t+1} can be depicted in terms of $(X_t, Y_{t+1} - Y_t)$. In more general circumstances we would incorporate a solution to a a filtering problem if we are to match an information structure to (X, Y), a filtering problem that is perhaps solved by economic agents.

Consistent intertemporal pricing together with the Markov property lead us to use a class of stochastic processes called multiplicative functionals. These processes are built from the underlying Markov process and will be used to model cash flows and stochastic discount factors. Since many macroeconomic time series grow or decay over time, we use the state vector X to model the growth

4

rate of such processes. In particular, let the dynamics of a *multiplicative functional* M be defined as[1]

$$\log M_{t+1} - \log M_t = \kappa\left(X_t, W_{t+1}\right). \tag{2}$$

The components of Y are examples of multiplicative functionals. Since X is stationary, the process $\log M$ has stationary increments. A revealing example is the conditionally linear model

$$\kappa\left(X_t, X_{t+1}\right) = \beta\left(X_t\right) + \alpha\left(X_t\right) \cdot W_{t+1}$$

where $\beta\left(x\right)$ allows for nonlinearity in the conditional mean, and $\alpha\left(x\right)$ introduces stochastic volatility.

We denote G a generic cash-flow process and S the equilibrium determined stochastic discount factor process, both modeled as multiplicative functionals. While we adopt a common mathematical formulation for both, G is expected to grow and S is expected to decay over time, albeit in stochastic manners.

Equilibrium models in macroeconomics and asset pricing build on the premise of utility-maximizing investors trading in arbitrage-free markets. Arbitrage-free pricing implies the existence of a strictly positive stochastic discount factor process S that can be used to infer equilibrium asset prices. Stochastic discount factors provide a convenient way to depict the observable implications of asset pricing models.[2] In this chapter, we consider a stochastic discount factor process that compounds the one-period stochastic discount factors in order to value multiperiod claims.

Definition 2.1. *A stochastic discount factor S is a positive (with probability one) stochastic process such that for any $t, j \geq 0$ and payoff G_{t+j} maturing at time $t + j$, the time-t price is given by*

$$\mathcal{Q}_t\left[G_{t+j}\right] = E\left[\left(\frac{S_{t+j}}{S_t}\right) G_{t+j} \mid \mathcal{F}_t\right]. \tag{3}$$

Notice that this definition does not restrict the date zero stochastic discount factor, S_0. This initialization may be chosen in a convenient manner. If markets are complete, then this stochastic discount factor is unique up to the initialization. Equations of the type (3) arise from investors' optimality conditions in the form of Euler equations. In an equilibrium model with complete markets, the stochastic discount factor is typically equated with the marginal rate of substitution of an unconstrained investor. The identity of such a person can change over time and across states. In some models with incomplete markets, the stochastic discount factor process ceases to be unique. There are different shadow prices for non-traded risk exposures but a common pricing of the exposures with explicit compensations in financial markets. With other forms of trading frictions, the pricing equalities can be replaced by pricing inequalities, still expressed using a stochastic discount factor.

In our framework, we will suppose that equilibrium stochastic discount factors inherit the

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[1]Multiplicative functionals are often initialized to be at one, or equivalently $\log M_0 = 0$. We will abuse this jargon a bit by allowing ourselves other possible initiations.

[2]See Hansen and Richard (1987) for an initial discussion of stochastic discount factors.

multiplicative functional structure. Market frictions, portfolio constraints and other types of market imperfections will then introduce distortions into formula (3). We will study such distortions in models with financial constraints in Section 7.

Notice that definition (3) of the stochastic discount factor involves an expectations operator. This expectations operator in general represents investors' beliefs about the future. Here, we have imposed rational expectations by assuming that investors' beliefs are identical to the data-generating probability measure P. This measure is that implied by historical evidence or by the fully specified model. Investors' beliefs, however, may differ from P and there exists alternative approaches to modeling these deviations in interesting ways. While the modeling of investors' beliefs is an important building block of the asset pricing framework, in this chapter we abstract from these considerations and impose rational expectations throughout the text.

3   Asset pricing over alternative investment horizons

We price cash flows exposed to macroeconomic uncertainty and modeled as multiplicative processes. Consider a generic cash flow process G, say the dividend process or an equilibrium consumption process. We start with a baseline payoff G_t maturing in individual periods $t = 0, 1, 2, \ldots$ and parameterize stochastic perturbations of this process. In particular, we derive measures that capture the sensitivity of expected payoff to exposure to alternative macroeconomic shocks, and the sensitivity of the associated risk compensations. We follow the convention in empirical finance by depicting compensations in terms of expected returns per unit of some measure of riskiness. The compensations differ depending upon which shock we target when we construct stochastic perturbations. The method relies on a comparison of the pricing of payoff G_t relative to another payoff that is marginally more exposed to risk in a particular way.

The cash flows G arising from equilibrium models will often have the form of multiplicative processes (2). A special case of such cash flows are payoffs that are positive functions of the Markov state, $\psi(X_t)$. These payoffs will be featured prominently in our subsequent analysis.

3.1   One-period pricing

We are interested in the pricing of payoffs maturing at different horizons but we start with a simple one-period conditionally lognormal environment. This environment will provide an explicit link to familiar calculations in asset valuation. Suppose that

$$
\begin{aligned}
\log G_1 &= \beta_g(X_0) + \alpha_g(X_0) \cdot W_1 \\
\log S_1 - \log S_0 &= \beta_s(X_0) + \alpha_s(X_0) \cdot W_1
\end{aligned}
$$

where G_1 is the payoff to which we assign values and S_1 is the one-period stochastic discount factor used to compute these values. The one-period return on this investment is the payoff in period one

divided by the period-zero price:

$$R_1 \doteq \frac{G_1}{Q_0\,[G_1]} = \frac{\left(\frac{G_1}{G_0}\right)}{E\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) \mid X_0\right]}.$$

The logarithm of the expected return can then be calculated explicitly as:

$$
\begin{aligned}
\log E\,[R_1 \mid X_0 = x] &= \log E\left[\left(\frac{G_1}{G_0}\right) \mid X_0 = x\right] - \log E\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) \mid X_0 = x\right] \qquad (4)\\
&= \underbrace{-\beta_s(x) - \frac{|\alpha_s(x)|^2}{2}}_{\text{risk-free rate}} \underbrace{- \alpha_s(x) \cdot \alpha_g(x).}_{\text{risk premium}}
\end{aligned}
$$

This compensation is expressed in terms of expected returns as is typical in asset pricing. Notice that we are using logarithms of proportional risk premia as a starting point.

Imagine applying this calculation to a family of such payoffs parameterized in part by α_g. The vector α_g defines a vector of exposures to the components of the normally distributed shock W_1. Then $-\alpha_s$ is the vector of shock "prices" representing the compensation for exposure to these shocks.

The risk prices in this conditionally lognormal model have a familiar conditional linear structure known from one-period factor models. In these models, the so called factor loadings α_g on the individual shocks W_1 are multiplied by factor prices $-\alpha_s$. The total compensation in terms of an expected return is thus the product of the quantity of risk (risk exposure), and the price per unit of this risk. There are analogous simplifications for continuous-time diffusion models since the local evolution in such models is conditionally normal.

In a nonlinear multiperiod environment, this calculation ceases to be straightforward. We would, however, still like to infer measures of the quantity of risk and the associated price of the risk. We therefore explore a related derivation that will yield the same results in this one-period lognormal environment but will also naturally extend to a nonlinear setup and multiple-period horizons.

3.1.1 One-period shock elasticities

We parameterize a family of random variables $H_1(\mathsf{r})$ indexed by r using

$$\log H_1(\mathsf{r}) = \mathsf{r}\nu(X_0) \cdot W_1 - \frac{\mathsf{r}^2}{2}|\nu(X_0)|^2 \qquad (5)$$

where r is an auxiliary scalar parameter. The vector of exposures $\alpha_h(X_0)$ is normalized to

$$E\left[|\nu(X_0)|^2\right] = 1.$$

With this normalization,

$$E\,[H_1(\mathsf{r})\,|X_0] = 1.$$

Even when shocks are not normally distributed, we shall find it convenient to construct $H_1(\mathsf{r})$ to have a unit conditional expectation.

Given the baseline payoff G_t, form a parameterized family of payoffs $G_1 H_1(\mathsf{r})$ given by

$$\log G_1 - \log G_0 + \log H_1(\mathsf{r}) = \underbrace{[\alpha_g(X_0) + \mathsf{r}\nu(X_0)]}_{\text{new shock exposure}} \cdot W_1 + \beta_g(X_0) - \frac{\mathsf{r}^2}{2}|\nu(X_0)|^2.$$

The new cash flow $G_1 H_1(\mathsf{r})$ has shock exposure $\alpha_g(X_0) + \mathsf{r}\nu(X_0)$ and is thus more exposed to the vector of shocks W_1 in the direction $\nu(X_0)$. By changing r we alter the magnitude of the exposure in direction $\nu(X_0)$. By choosing different vectors $\nu(X_0)$, we alter the combinations of shocks whose impact we want to investigate. A typical example of an $\nu(X_0)$ would be a coordinate vector e_j with a single one in j-th place. In that case, we infer the pricing implications of the j-the component of the shock vector W_1. In some applications it may be convenient to make $\nu(X_0)$ explicitly depend on X_0. For instance, Borovička et al. (2011) propose scaling of ν with X_0 in models with stochastic volatility.

The payoffs $G_1 H_1(\mathsf{r})$ imply a corresponding family of logarithms of expected returns as in equation (4):

$$\log E[R_1(\mathsf{r}) \mid X_0 = x] = \log E\left[\left(\frac{G_1}{G_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right] - \log E\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right].$$

We are interested in comparing the expected return of the payoff $G_1 H_1(\mathsf{r})$ relative to $G_1 = G_1 H_1(0)$. Since our exposure direction $\nu(X_0)$ has a unit standard deviation, by differentiating with respect to r we compute an elasticity

$$\frac{d}{d\mathsf{r}} \log E[R_1(\mathsf{r}) \mid X_0 = x]\big|_{\mathsf{r}=0}$$
$$= \frac{d}{d\mathsf{r}} \log E\left[\left(\frac{G_1}{G_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right]\bigg|_{\mathsf{r}=0} - \frac{d}{d\mathsf{r}} \log E\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right]\bigg|_{\mathsf{r}=0}.$$

This elasticity measures the sensitivity of the expected return on the payoff G_1 to an increase in exposure to the shock in the direction $\nu(x)$. The calculation leads us to define the counterparts of quantity and price elasticities from microeconomics:

1. The one-period *shock-exposure elasticity*

$$\varepsilon_g(x,1) = \frac{d}{d\mathsf{r}} \log E\left[\left(\frac{G_1}{G_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right]\bigg|_{\mathsf{r}=0} = \alpha_g(x) \cdot \nu(x)$$

measures the sensitivity of the expected payoff G_1 to an increase in exposure in the direction $\nu(x)$.

2. The one-period *shock-price elasticity*

$$
\begin{aligned}
\varepsilon_p\left(x,1\right) &= \left.\frac{d}{d\mathsf{r}}\log E\left[\left(\frac{G_1}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right]\right|_{\mathsf{r}=0} - \left.\frac{d}{d\mathsf{r}}\log E\left[\left(\frac{S_1}{S_0}\right)\left(\frac{G_1}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right]\right|_{\mathsf{r}=0} \\
&= -\alpha_s\left(x\right)\cdot\nu\left(x\right)
\end{aligned}
$$

measures the sensitivity of the compensation, in units of expected return, for this exposure.

Notice that the shock-exposure elasticity recovers the exposure vector $\alpha_g\left(x\right)$, and individual components of this vector can be obtained by varying the choice of the direction of the perturbation $\nu\left(x\right)$. Similarly, the shock-price elasticity recovers the vector of prices $-\alpha_s\left(x\right)$ associated with the risks embedded in the shock W_1.

In this one-period case, we replicated a straightforward decomposition of the expected return (4) into quantities and prices of risk. Now we move to the characterization of the asset pricing implications over longer horizons.

3.2 Multi-period investment horizon

Consider the parameterized payoff $G_t H_1\left(\mathsf{r}\right)$ with a date-zero price $E\left[S_t G_t H_1\left(\mathsf{r}\right)\mid X_0 = x\right]$. This is a payoff maturing at time t that has the same growth rate as payoff G_t except period one when the growth rate is stochastically perturbed by $H_1\left(\mathsf{r}\right)$. The logarithm of the expected return (yield to maturity) is

$$
\log E\left[R_{0,t}\left(\mathsf{r}\right)\mid X_0 = x\right] \doteq \log E\left[\left(\frac{G_t}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right] - \log E\left[\left(\frac{S_t}{S_0}\right)\left(\frac{G_t}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right].
$$

Following our previous analysis, we construct two elasticities:

1. *shock-exposure elasticity*

$$
\varepsilon_g\left(x,t\right) = \left.\frac{d}{d\mathsf{r}}\log E\left[\left(\frac{G_t}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right]\right|_{\mathsf{r}=0}
$$

2. *shock-price elasticity*

$$
\varepsilon_p\left(x,t\right) = \left.\frac{d}{d\mathsf{r}}\log E\left[\left(\frac{G_t}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right]\right|_{\mathsf{r}=0} - \left.\frac{d}{d\mathsf{r}}\log E\left[\left(\frac{S_t}{S_0}\right)\left(\frac{G_t}{G_0}\right)H_1\left(\mathsf{r}\right)\mid X_0 = x\right]\right|_{\mathsf{r}=0}
\tag{6}
$$

These elasticities are functions of the investment horizon t, and thus we obtain a term structure of elasticities. The dependence on the current state $X_0 = x$ incorporates possible time-variation in the sensitivity of expected returns to exposure to shocks.

3.3 A change of measure and an impulse response for a multiplicative functional

Notice that the shock elasticities defined in the previous subsection have a common mathematical structure expressed using the multiplicative functionals $M = S$ and $M = SG$. Given a multiplicative functional M, we define

$$\varepsilon(x,t) = \frac{d}{d\mathsf{r}} \log E\left[\left(\frac{M_t}{M_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right]\Bigg|_{\mathsf{r}=0} \tag{7}$$

Taking the derivative in (7), we obtain

$$\varepsilon(x,t) = \nu(x) \cdot \frac{E\left[\left(\frac{M_t}{M_0}\right) W_1 \mid X_0 = x\right]}{E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]} \tag{8}$$

Thus a major ingredient in the computation is the covariance between $\left(\frac{M_t}{M_0}\right)$ and W_1 conditioned on X_0.

The random variable $H_1(r)$ given by (5) is positive and has expectation equal to unity conditioned on X_0. Multiplication by the this random variable has the interpretation of changing the probability distribution of W_1 from having mean zero to having a mean given by $\mathsf{r}\nu(X_0)$. Thus given a multiplicative process M

$$E\left[\left(\frac{M_t}{M_0}\right) H_1(\mathsf{r}) \mid X_0 = x\right] = E\left[H_1(\mathsf{r}) E\left[\left(\frac{M_t}{M_0}\right) \mid X_0, W_1 \mid X_0\right]\right]$$
$$= \widetilde{E}\left(E\left[\left(\frac{M_t}{M_0}\right) \mid X_0, W_1\right] \mid X_0 = x\right)$$

where $\widetilde{E}$ presumes that the random vector W_1 is distributed as a multivariate normal with mean $\mathsf{r}\nu(x)$ consistent with our multiplication by $H_1(\mathsf{r})$.

3.4 Long-horizon pricing

Shock elasticities depict the term structure of risk as we change the maturity of priced payoffs. To aid our understanding of the overall shape of the term structure of elasticities, we characterize the long-horizon limits of these shock elasticities. We provide a characterization for a general multiplicative process that takes the form of a factorization. A multiplicative process is a product of a geometric constant growth or decay process, a positive martingale and a ratio of a function of the Markov state in date zero and date t. Since the factorization is applicable to any member of a general class of multiplicative processes, we apply it to both stochastic discount factor processes and positive cash flow processes.

As in Hansen and Scheinkman (2009) and Hansen (2012), we use Perron–Frobenius theory to provide a factorization of multiplicative processes. Given a multiplicative process M, solve the

equation

$$E\left[\left(\frac{M_t}{M_0}\right)e\left(X_t\right)\mid X_0 = x\right] = \exp\left(\eta t\right)e\left(x\right) \tag{9}$$

for an unknown function $e\left(x\right)$ that is strictly positive and an unknown number η. The solution is independent of the choice of the horizon t.

Consider the pair (e, η) that solves (9) and form

$$\frac{\widetilde{M}_t}{\widetilde{M}_0} \doteq \exp\left(-\eta t\right)\frac{e\left(X_t\right)}{e\left(X_0\right)}\left(\frac{M_t}{M_0}\right). \tag{10}$$

The stochastic process $\widetilde{M}$ is a martingale under P, since

$$
\begin{aligned}
E\left[\widetilde{M}_{t+1}\mid\mathcal{F}_t\right] &= \frac{\exp\left[-\eta\left(t+1\right)\right]}{e\left(X_0\right)}\frac{M_t}{M_0}\widetilde{M}_0 E\left[\frac{M_{t+1}}{M_t}e\left(X_{t+1}\right)\mid\mathcal{F}_t\right] = \\
&= \exp\left(-\eta t\right)\frac{e\left(X_t\right)}{e\left(X_0\right)}\frac{M_t}{M_0}\widetilde{M}_0 = \widetilde{M}_t.
\end{aligned}
$$

Consequently, expression (10) can be reorganized as

$$\frac{M_t}{M_0} = \exp\left(\eta t\right)\frac{e\left(X_0\right)}{e\left(X_t\right)}\frac{\widetilde{M}_t}{\widetilde{M}_0} \tag{11}$$

This formula provides a multiplicative decomposition of the multiplicative functional M into a deterministic drift $\exp\left(\eta t\right)$, a stationary function of the Markov state $e\left(x\right)$ and a martingale $\widetilde{M}$. This martingale component will be critical in characterizing long-term pricing implications.

Associated with the martingale $\widetilde{M}$ is a probability measure $\widetilde{P}$ such that for every measurable function Z of the Markov process between dates zero and t,

$$E\left(\widetilde{M}_t Z\mid X_0 = x\right) = \widetilde{E}\left(Z\mid X_0 = x\right)$$

where $\widetilde{E}\left(\cdot\mid X_0 = x\right)$ is the conditional expectation operator under the probability measure $\widetilde{P}$.[3]

In finite state spaces, equation (9) can be posed as a matrix problem with a solution that is an eigenvector with positive entries.

Example 3.1. *In a finite-state Markov chain environment, equation* (9) *is a standard eigenvalue problem. Let realized value of the X_t be represented as alternative coordinate vectors. Suppose the ratio $\frac{M_{t+1}}{M_t}$ satisfies*

$$\frac{M_{t+1}}{M_t} = \left(X_{t+1}\right)'\mathbf{M}X_t$$

for some square matrix $\mathbb{M}$. In the same way, represent the one-period transition probabilities as a

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[3] In order to completely define the measure $\widetilde{P}$, we also need to specify the unconditional probability distribution. For instance, $\widetilde{M}_0$ can be initiated to make $\widetilde{P}$ stationary. Since all pricing results in this chapter utilize conditional probability distributions, we abstract from these considerations here.

matrix **P**. *For* $t = 1$, *equation* (9) *becomes a vector equation*

$$(\mathbf{P} * \mathbf{M}) e = \exp(\eta) e$$

where the operator $*$ *depicts elementwise multiplication,* $(\mathbf{P} * \mathbf{M})_{ij} = \mathbf{P}_{ij}\mathbf{M}_{ij}$. *When*

$$\sum_{j=0}^{\infty} \lambda^j (\mathbf{P} * \mathbf{M})^j$$

has all strictly positive entries for some $0 < \lambda < 1$, *the Perron–Frobenius theorem implies the existence of a unique normalized strictly positive eigenvector* **e** *associated with the largest eigenvalue* $\exp(\eta)$ *of the matrix* $\mathbf{P} * \mathbf{M}$. *Then* $e(X_t)$ *in formula* (9) *is* $\mathbf{e} \cdot X_t$.

In continuous state spaces, this factorization may not yield a unique strictly positive solution $e(x)$. Hansen and Scheinkman (2009) and Borovička et al. (2015) provide selection criteria based on the stochastic stability of the probability measure implied by the martingale component to guarantee uniqueness. Stochastic stability ensures that we have a valuable way to compute limiting approximations once we change measures. Here, we will assume that we have selected such a solution.[4]

Factorization (11) leads to a characterization of long-horizon limits for the shock elasticities. Using this factorization in expression (7), we obtain

$$\varepsilon(x, t) = \nu(x) \cdot \frac{\widetilde{E}\left[\widehat{e}(X_t) W_1 \mid X_0 = x\right]}{\widetilde{E}\left[\widehat{e}(X_t) \mid X_0 = x\right]}$$

where $\widehat{e}(x) \doteq 1/e(x)$. Under technical assumptions, we find that[5]

$$\lim_{t \to \infty} \varepsilon(x, t) = \nu(x) \cdot \widetilde{E}\left[W_1 \mid X_0 = x\right].$$

Factorization (11) leads to a characterization of long-horizon limits for the shock elasticities. Using this factorization in expression (7), we obtain

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where $\widehat{e}(x) \doteq 1/e(x)$. Under technical assumptions the long-maturity limit for the shock elasticity is given by

$$\lim_{t \to \infty} \varepsilon(x, t) = \nu(x) \cdot \widetilde{E}\left[W_1 \mid X_0 = x\right].$$

The sensitivity of long-horizon payoffs to current shocks is therefore determined by the martingale components of the stochastic discount factor and the cash flow, and their implications for the

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[4]Our formulation presumes an underlying Markovian structure. See Qin and Linetsky (2014b) for a more general starting point and an analogous factorization.

[5]See Hansen and Scheinkman (2012) for a version of this result for a continuous-time diffusion model.

expectations of shock W_1 as captured by the implied change in probability measures.

3.5   Non-Gaussian frameworks

While we have made special reference to normally distributed shocks, our mathematical structure does not require this. We have featured perturbations $H_1(\mathsf{r})$ that are positive and expectations one. Risk prices in financial economics are denominated in terms of expected mean compensation per unit of risk. With normally distributed shocks, we measure risk in units of standard deviations. Provided that we adopt an interpretable way to denominate risk prices for other distributions, our methods continue to apply beyond the conditionally Gaussian framework. For instance, Zviadadze (2016) constructs shock elasticities in a stochastic environment with autoregressive gamma processes.

Another example are regime-shift models that may include both normally distributed shocks along with uncertain regimes. Exposure to macroeconomic regime-shift risk is of interest and can be characterized using shock elasticities by structuring appropriately the random variable $H_1(\mathsf{r})$. These switches can be exogenous (e.g., exogenously modeled periods of low or high growth and volatility) or endogenous (e.g., interest rate at the zero lower bound, financial sector in a period of binding financial constraints, or regime changes in government policies). We develop shock elasticities for regime-shift risk in Borovička et al. (2011).

For Markov chain models used to capture the regime shift dynamics of exogenous shocks see David (2008), Chen (2010) or Bianchi (2015) for some recent examples in the asset pricing literature and Liu et al. (2011) and Bianchi et al. (2013) in macroeconomic modeling. Regime switches are also utilized to model time-variation in government policies, see Sims and Zha (2006), Liu et al. (2009) and Bianchi (2012) for regime switching in monetary policy rules, Davig et al. (2010, 2011) and Bianchi and Melosi (2015) for fiscal policy applications and Chung et al. (2007) and Bianchi and Ilut (2015) for a combination of both. Farmer et al. (2011) and Foerster et al. (2014) analyze solution and estimation techniques in Markov chain models in conjunction with perturbation approximation methods. In Borovička and Hansen (2014), we introduce a tractable exponential-quadratic framework that permits semi-analytical formulas for shock elasticities and encompasses a large class of models solved using perturbation techniques.

4   Relation to impulse response functions

Impulse responses to specific structural shocks are a common way of representing the dynamic properties of macroeconomic models. As we mentioned previously, this idea goes back at least to Frisch (1933). Our elasticity computations change exposures of cash flows to shocks and explore the consequences for valuation. These constructs are closely related and in some circumstances are mathematically identical to impulse response functions. We explore these connections in this section.

To relate our elasticity calculation to an impulse response function, consider the conditional

expectation

$$E\left[\left(\frac{M_t}{M_0}\right) \mid X_0, W_1 = w\right]$$

for alternative choices of w. Changing the value of w gives rise to the impulse response of M_t to a shock at date one. Instead of conditioning on alternative realized values of the shock at date one, as we have seen our computations are equivalent to changing the date zero distribution of W_1. A similarity in perspectives emerges because this distributional change could include a mean shift in the distribution for W_1. In practice, empirical macroeconomists typically study expectations of the logarithms of macroeconomic time series, often using linear models. For asset pricing it is important that we work with expectations of levels of macroeconomic quantities and cash flows, and account for nonlinearities. To compute shock *elasticities* we are lead to study the impact on the logarithm of the conditional expectation of M_t as developed in formula (7). In the remainder of this section, we consider two special cases in which the link to impulse functions is particularly close.

4.1 Lognormality

When M is a lognormal process, the impulse response functions for $\log M$ matches exactly our shock elasticity as we will now see.

A linear vector-autoregression (VAR) model is special case of the framework (1). Specifically X is a linear vector-autoregression with autoregression coefficient matrix $\bar{\mu}$ and shock exposure matrix $\bar{\sigma}$:

$$X_{t+1} = \bar{\mu}X_t + \bar{\sigma}W_{t+1}. \tag{12}$$

We assume that the absolute values of eigenvalues of the matrix $\bar{\mu}$ are strictly less than one. Analogously, we introduce a multiplicative process M (constructed in general form in (2)) with evolution:

$$\log M_{t+1} - \log M_t = \bar{\beta} \cdot X_t + \bar{\alpha} \cdot W_{t+1}. \tag{13}$$

The shock W_{t+1} is distributed as a multivariate standard normal. With this construction of the multiplicative process M, we first study the responses of $\log M$.

4.1.1 Impulse response functions

Let $\nu(x) = \bar{\nu}$ where $\bar{\nu}$ is a vector with norm one. In typical applications, $\bar{\nu}$ is a coordinate vector. The impulse response function of Y_t for the linear combination of shocks chosen by the vector $\bar{\nu}$ is given by

$$E\left[\log M_t - \log M_0 \mid X_0 = x, W_1 = \bar{\nu}\right] - E\left[\log M_t - \log M_0 \mid X_0 = x, W_1 = 0\right] = \bar{\nu} \cdot \bar{\varrho}_t.$$

where the coefficients satisfy the recursions implied by (12) and (13). From (13), we have the recursion:

$$\bar{\varrho}_{t+1} - \bar{\varrho}_t = \left(\bar{\zeta}_t\right)' \bar{\beta} \tag{14}$$

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with initial condition $\bar{\varrho}_1 = \bar{\alpha}$, and from (12):

$$\bar{\zeta}_{t+1} = \bar{\mu}\bar{\zeta}_t \tag{15}$$

with initial condition $\bar{\zeta}_1 = \bar{\sigma}$. Solving these recursions gives:

$$\bar{\zeta}_t = \bar{\mu}^{t-1}\bar{\sigma} \tag{16}$$

$$\bar{\varrho}_t = \bar{\alpha} + \left[(I - \bar{\mu})^{-1} \left(I - \bar{\mu}^{t-1} \right) \bar{\sigma} \right]' \bar{\beta}.$$

The impulse response function in the linear model is thus a sequence of deterministic coefficients $\bar{\nu} \cdot \bar{\varrho}_t$. The first term, $\bar{\alpha} \cdot \bar{\nu}$, represents the immediate response arising from realization $\bar{\nu}$ of the current shock while the remaining terms capture the subsequent propagation of the shock through the dynamics of state vector X as it influences $\log M$ in the future.

4.1.2   Shock elasticities

Consider now our elasticity calculation. Write $\log M_t$ as its moving-average representation:

$$\log M_t = \sum_{j=0}^{t-1} \bar{\varrho}_j \cdot W_{t-j} + E\left(\log M_t \mid \mathcal{F}_0 \right),$$

or equivalently

$$\log M_t - \log M_0 = \sum_{j=1}^{t} \bar{\varrho}_j \cdot W_{t-j+1} + E\left(\log M_t - \log M_0 \mid X_0 \right)$$

$$= \sum_{j=1}^{t-1} \bar{\varrho}_j \cdot W_{t-j+1} + \bar{\varrho}_t \cdot W_1 + E\left(\log M_t - \log M_0 \mid X_0 \right).$$

Since the shocks W_t are independently distributed as a multivariate standard normals over time,

$$E\left[\left(\frac{M_t}{M_0} \right) \mid X_0 = x, W_1 = w \right] = \exp\left(\frac{1}{2} \sum_{j=1}^{t-1} \bar{\varrho}_j \cdot \bar{\varrho}_j \right) \exp\left(\bar{\varrho}_t \cdot W_1 \right) \exp\left(E\left[\log M_t - \log M_0 \mid X_0 \right] \right)$$

Using formula (8), we compute:

$$\varepsilon(x, t) = \frac{E\left[\left(\frac{M_t}{M_0} \right) W_1 \mid X_0 = x \right]}{E\left[\left(\frac{M_t}{M_0} \right) \mid X_0 = x \right]} = \frac{E\left[\exp\left(\bar{\varrho}_t \cdot W_1 \right) W_1 \mid X_0 = x \right]}{E\left[\exp\left(\bar{\varrho}_t \cdot W_1 \right) \right]} = \bar{\varrho}_t$$

The second equality follows by observing that

$$\frac{\exp\left(\bar{\varrho}_t \cdot W_1 \right)}{E\left[\exp\left(\bar{\varrho}_t \cdot W_1 \right) \right]}$$

is strictly positive and has conditional expectation one. Multiplication by this random variable is equivalent to changing the distribution of W_1 from a multivariate standard normal to a multivariate normal with mean $\bar{\varrho}_t$. To summarize, in this lognormal case, the shock elasticities do not depend on the Markov state and they coincide with the impulses responses measured by $\bar{\nu} \cdot \bar{\varrho}_t$ for $t = 1, 2, \ldots$.

Consider in particular the shock-price elasticity (6). Notice that this shock-price elasticity consists of the difference of shock elasticities for G and SG, and thus we are lead to compute impulse response functions for $\log G$ and $\log S + \log G$. The additivity of the construction implies that the impulse response function coefficients for the latter are $\bar{\nu} \cdot \bar{\varrho}_{s,t} + \bar{\nu} \cdot \bar{\varrho}_{g,t}$, and thus the resulting shock-price elasticity corresponds to the impulse response function of $-\log S$, with coefficients $-\bar{\nu} \cdot \bar{\varrho}_{s,t}$.

4.1.3 Long-term pricing revisited

In this example, as discussed in Hansen et al. (2008) there is a close link between the factorization described in Section 3.4 and the additive decompositions of linear time series. Beveridge and Nelson (1981) and Blanchard and Quah (1989) extracted a martingale component in linear models and used it to characterize the impact of permanent shocks.[6]

Consider solving

$$E\left[\left(\frac{M_1}{M_0}\right) e\left(X_1\right) \mid X_0 = x\right] = \exp\left(\eta\right) e\left(x\right)$$

for the pair (e, η), where the evolution of M is given by (13). In this special case, a straightforward calculation using formulas for lognormals gives:

$$\log e(x) = E\left(\sum_{j=0}^{\infty} \bar{\beta} \cdot X_{t+j} \mid X_t = x\right)$$
$$= (\bar{\beta})'(I - \bar{\mu})^{-1} x,$$

and[7]

$$\eta = \frac{1}{2}|\bar{\alpha}' + \bar{\beta}'(I - \bar{\mu})^{-1}\bar{\sigma}|^2.$$

Under the change of measure associated with the martingale $\widetilde{M}$ in the multiplicative factorization, W_1 has a mean equal to

$$\bar{\sigma}'(I - \bar{\mu}')^{-1}\bar{\beta} + \bar{\alpha}$$

which is independent of the state vector. Notice that this is also the limiting value of $\bar{\varrho}_t$ as given in (16). In this lognormal example

$$\log M_{t+1} - \log M_t + \log e(X_{t+1}) - \log e(X_t) = \left[\bar{\beta}'(I - \bar{\mu}')^{-1}\bar{\sigma} + \bar{\alpha}'\right] W_{t+1}$$

---

[6] Hansen (2012) constructs an additive decomposition of $\log M$ in a continuous-time version of our nonlinear framework.

[7] If we were to include a constant included in the evolution of $\log M$, this would be added to η.

where the right-hand side gives the permanent shock to $\log M$ as constructed in Beveridge and Nelson (1981) and Blanchard and Quah (1989). In VAR analyses, transitory shocks are typically constructed as linear combinations of W_{t+1} that are uncorrelated with this permanent shock. On the other hand $\log e(X_{t+1})$ and its innovation are typically correlated with the permanent shock.

This simple connection between permanent shocks and permanent components to pricing ceases to hold in more general nonlinear environments. Hansen (2012) has a more complete discussion of the relation between the permanent component to $\log M$ and the martingale component to M outside this lognormal specification.

4.2 Continuous-time diffusions

In this subsection, we focus on a framework with uncertainty modeled using Brownian shocks, and apply it to models with financial constraints in Section 7. While the Brownian information setup is not without loss of generality, it provides tools for a pedagogically transparent treatment and shows the close connection between shock elasticities and impulse responses. In Borovička et al. (2011) we also consider jumps in the form of regime shifts in continuous-time Markov chains and applications to consumption-based asset pricing models.

Let X be a Markov diffusion on $\mathcal{X} \subseteq \mathbb{R}^n$:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

with initial condition $X_0 = x$. Here, $\mu(x)$ is an n-dimensional vector and $\sigma(x)$ is an $n \times k$ matrix for each vector x in $\mathbb{R}^n$. In addition W is a k-dimensional Brownian motion. We use this underlying Markov process to construct a multiplicative process M via:

$$\log M_t = \log M_0 + \int_0^t \beta(X_u)du + \int_0^t \alpha(X_u) \cdot dW_u \tag{17}$$

where $\beta(x)$ is a scalar and $\alpha(x)$ is a k-dimensional vector, or, in differential notation,

$$d\log M_t = \beta(X_t)\,dt + \alpha(X_t) \cdot dW_t. \tag{18}$$

Thus M_t depends on the initial conditions $(X_0, M_0) = (x, m)$ and the innovations to the Brownian motion W between dates zero and t. Let $\{\mathcal{F}_t : t \geq 0\}$ be the (completed) filtration generated by the Brownian motion between time zero and time t along with any initial information captured by $\mathcal{F}_0$.

As before, stochastic discount factors and cash flows in this environment are specific versions of a multiplicative process M. This multiplicative process is exposed to two types of risk. The first source of risk exposure is the 'local', or infinitesimal, risk in term $\alpha(X_u) \cdot dW_u$ in (17). The second source of risk comes from the time-variation in X_t and the state dependence of coefficients $\beta(x)$ and $\alpha(x)$, and is manifested over longer horizons.

4.2.1 Haussmann–Clark–Ocone formula

There is a natural counterpart to a moving-average representation for diffusions. Importantly, the moving-average coefficients are, in general, state dependent. They entail computing so-called Malliavin derivatives of the date-u shock to the process $\log M_t$ for $t \geq u$, denoted $\mathcal{D}_u \log M_t$. We do not develop Malliavin differentiation as a formal mathematical construct but instead proceed heuristically.[8] This calculation of a Malliavin derivative gives the random response to a shock at date u and is only restricted to be t-measurable where $t \geq u$. By forming the date-u conditional expectation we get the expected response as of the date of the shock. The computation is localized by making the time interval over which the shock acts on the process $\log M_t$ arbitrarily small, which allows for the formal construction of a derivative.

The calculation of $\mathcal{D}_u \log M_t$ has two uses analogous to the lognormal example we examined earlier. First, the (random) impulse response function for $\log M$

$$\varrho_t(X_0) = \nu(X_0) \cdot E\left(\mathcal{D}_0 \log M_t \mid F_0\right) = \nu(X_0) \cdot E\left[\mathcal{D}_0 \left(\log M_t - \log M_0\right) \mid X_0\right]$$

for $t \geq 0$ where $\nu(X_0)$ determines which conditional linear combination of the shocks is subject to an impulse. The resulting responses depend on conditioning information captured by X_0, in contrast to lognormal models in which responses depend only on the horizon $t \geq 0$. Relatedly we obtain the Haussmann–Clark–Ocone formula for the process $\log M$ that cumulates the impact shocks at various dates as a stochastic integral:

$$\log M_t = \int_0^t E\left(\mathcal{D}_u \log M_t \mid F_u\right) \cdot dW_u + E\left(\log M_t \mid \mathcal{F}_0\right).$$

where we may think of $E\left(\mathcal{D}_u \log M_t \mid F_u\right)$ as the counterpart to a coefficient vector in a moving-average representation. These random variables satisfy recursions analogous to (14) and (15). For a more detailed construction, see Borovička et al. (2014).

We use the rules of Malliavin differentiation (analogous to more familiar forms of differentiation):

$$\mathcal{D}_u M_t = M_t \mathcal{D}_u \log M_t,$$

implying that the impulse response function for the process M is

$$\nu(X_0) \cdot E\left(\mathcal{D}_0 M_t \mid \mathcal{F}_0\right) = \nu(X_0) \cdot E\left(M_t \mathcal{D}_0 \log M_t \mid \mathcal{F}_0\right)$$
$$= M_0 \nu(X_0) \cdot E\left[\left(\frac{M_t}{M_0}\right) \mathcal{D}_0 \left(\log M_t - \log M_0\right) \mid X_0\right)$$

for $t \geq 0$.

---

[8]For a textbook treatment of Malliavin calculus see Di Nunno and Øksendal (2009) or Nualart (2006).

4.2.2 Shock elasticities for diffusions

The construction of shock elasticities in Section 3 perturbs the cash flow by exposing it to a specified shock in the next period. In the continuous-time model, we devise a perturbation of M over a short time interval $[0, r]$ and then study the implications as $r \searrow 0$. The resulting construction exploits the local linearity of continuous-time models with Brownian shocks.

Specifically, we construct the process H^r such that

$$\log H_t^r = \int_0^{r \wedge t} \nu(X_u) \cdot dW_u - \frac{1}{2} \int_0^{r \wedge t} |\nu(X_u)|^2 \, du.$$

where $r \wedge t = \min\{r, t\}$. Notice that this process is exposed to the Brownian shock on the time interval $[0, r]$, with exposure vector $\nu(x)$, and stays constant after r. We assume that $\nu(x)$ is restricted so that the process H^r is a martingale. We use H^r to construct the perturbed process MH^r:

$$
\begin{aligned}
\log M_t + \log H_t^r &= \log M_0 + \int_0^t \beta(X_u) du - \frac{1}{2} \int_0^{r \wedge t} |\nu(X_u)|^2 \, du \\
&\quad + \int_0^t \alpha(X_u) \cdot dW_u + \int_0^{r \wedge t} \nu(X_u) \cdot dW_u
\end{aligned}
$$

Notice that on the interval $[0, r]$, the exposure of the perturbed process to the Brownian shock is

$$[\alpha(X_u) + \nu(X_u)] \cdot dW_u.$$

As $r \searrow 0$, we are perturbing $\log M$ over an arbitrarily small interval.

As in Borovička et al. (2014), we define the shock elasticity for M at horizon t as

$$\varepsilon(x, t) = \lim_{r \searrow 0} \frac{1}{r} \log E\left[\left(\frac{M_t}{M_0}\right) H_t^r \mid X_0 = x\right]$$

and show that this limit can be expressed as

$$
\begin{aligned}
\varepsilon(x, t) &= \nu(x) \cdot \frac{E\left(\mathcal{D}_0 \frac{M_t}{M_0} \mid X_0 = x\right)}{E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]} \\
&= \nu(x) \cdot \frac{E\left[\left(\frac{M_t}{M_0}\right) \mathcal{D}_0 \log M_t \mid X_0 = x\right]}{E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]}.
\end{aligned} \tag{19}
$$

The first equality in (19) is a limiting version of (8) divided by $E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]$ since the Haussmann–Clark–Ocone formula applied to $\frac{M_t}{M_0}$ has a contribution

$$E\left(\mathcal{D}_0 \frac{M_t}{M_0} \mid X_0 = x\right) dW_0$$

for the date zero increment. The limiting covariance between $\frac{M_t}{M_0}$ and dW_0 is therefore $E\left(\left.\mathcal{D}_0 \frac{M_t}{M_0}\right| X_0 = x\right)$. From the second equality in (19), these elasticities coincide with the diffusion counterpart to impulse responses $\mathcal{D}_0(\log M_t - \log M_0)$ for $\log M_t - \log M_0$ weighted by

$$\frac{\left(\frac{M_t}{M_0}\right)}{E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]}$$

when averaging over future outcomes. For the lognormal model, the weighting is inconsequential. In Borovička et al. (2011), we provide details of this derivation and some related calculations including the following alternative formula relevent for computation:

$$\varepsilon(x,t) \doteq \nu(x) \cdot \left[\sigma(x)'\left(\frac{\partial}{\partial x}\log E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]\right) + \alpha(x)\right]. \tag{20}$$

The shock-elasticity formula (20) has a natural interpretation. The sensitivity of the multiplicative process M to a shock in the next instant consists of two terms. The term $\alpha(x)$ represents the direct impact of the Brownian shock on the evolution of M in expression (18). The partial derivative with respect to x capture the sensitivity of the conditional expectation to movements in the state vector, and it is multiplied by the exposure matrix $\sigma(x)$ to express the sensitivity with respect to the shock vector W. The use of the derivative of the logarithm in (18) justifies the term shock *elasticity*. The instantaneous short term elasticity is $\alpha(x) \cdot \nu(x)$.[9]

5   Discrete-time formulas and approximation

In the preceding sections, we developed formulas for shock-price and shock-exposure elasticities for a wide class of models driven by a state vector with Markov dynamics (1). We now present a tractable implementation that, when applicable, makes the computations straightforward to apply. The discussion draws on methods developed in Borovička and Hansen (2014).[10] We also provide Matlab software implementing the solution methods described in this section including a toolkit that computes shock elasticities for models solved using Dynare.[11]

We start by introducing a convenient exponential-quadratic framework that we use for modeling the state vector X and the resulting multiplicative processes. In this framework, conditional expectations of multiplicative processes and the shock elasticities are available in a convenient functional form. We then consider a special class of approximate solutions to dynamic macroeconomic models constructed using perturbation methods. We show how to approximate the equilibrium dynamics, additive and multiplicative functionals, and the resulting shock elasticities. By construction, the dynamics of these approximate solutions will be nested within the exponential-quadratic framework.

---

[9]The instantaneous shock-price elasticity is $-\alpha_s(x) \cdot \nu(x)$ which coincides with the notion of a risk price vector that represents the compensation for exposure to Brownian increments.

[10]See Nakamura et al. (2016) for another discrete-time implementation of these methods.

[11]Dynare is a freely available Matlab/Octave toolkit for solving and analyzing dynamic general equilibrium models. See http://www.dynare.org. Our software is available at http://borovicka.org/software.html.

5.1 Exponential-quadratic framework

We study dynamic systems for which the state vector can be partitioned as $X = (X_1', X_2')'$ where the two components follow the laws of motion:

$$
\begin{aligned}
X_{1,t+1} &= \Theta_{10} + \Theta_{11}X_{1,t} + \Lambda_{10}W_{t+1} \\
X_{2,t+1} &= \Theta_{20} + \Theta_{21}X_{1,t} + \Theta_{22}X_{2,t} + \Theta_{23}\left(X_{1,t} \otimes X_{1,t}\right) \\
&\quad + \Lambda_{20}W_{t+1} + \Lambda_{21}\left(X_{1,t} \otimes W_{t+1}\right) + \Lambda_{22}\left(W_{t+1} \otimes W_{t+1}\right).
\end{aligned}
\tag{21}
$$

We restrict the matrices Θ_{11} and Θ_{22} to have stable eigenvalues. Notice that the restrictions imposed by the triangular structure imply that the process X_1 is linear, while the process X_2 is linear conditional on the evolution of X_1.

The class of multiplicative functionals M that interest us satisfies, for $Y = \log M$, the restriction

$$
\begin{aligned}
Y_{t+1} - Y_t &= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) \\
&\quad + \Psi_0 W_{t+1} + \Psi_1\left(X_{1,t} \otimes W_{t+1}\right) + \Psi_2\left(W_{t+1} \otimes W_{t+1}\right).
\end{aligned}
\tag{22}
$$

In what follows we use a $1 \times k^2$ vector Ψ to construct a $k \times k$ symmetric matrix $\operatorname{sym}\left[\operatorname{mat}_{k,k}\left(\Psi\right)\right]$ such that[12]

$$
w'\left(\operatorname{sym}\left[\operatorname{mat}_{k,k}\left(\Psi\right)\right]\right)w = \Psi\left(w \otimes w\right).
$$

This representation will be valuable in some of the computations that follow. We use additive functionals to represent stochastic growth via a technology shock process or aggregate consumption, and to represent stochastic discounting used in representing asset values.

The system (21)–(22) is rich enough to accommodate stochastic volatility, which has been featured in the asset pricing literature and to a lesser extent in the macroeconomics literature. For instance, the state variable $X_{1,t}$ can capture a linear process for conditional volatility, and $X_{2,t}$ the conditional growth rate of cash flows. The coefficient Ψ_1 in (22) then determines the time-variation in the conditional volatility of the growth rate of M, while Λ_{21} in (21) impacts the conditional volatility of the changes in the growth rate. In Section 5.2, we will map the solution obtained using perturbation approximations into this framework as well.

A virtue of parameterization (21)–(22) is that it gives quasi-analytical formulas for our dynamic elasticities. The implied model of the stochastic discount factor has been used in a variety of reduced-form asset pricing models. Later we will use an approximation to deduce this dynamical system.

We illustrate the convenience of this functional form by calculating the logarithms of conditional expectations of multiplicative functionals of the form (22). Consider a function that is linear-

---

[12] In this formula $\operatorname{mat}_{k,k}\left(\Psi\right)$ converts a vector into a $k \times k$ matrix and the sym operator transforms this square matrix into a symmetric matrix by averaging the matrix and its transpose. Appendix A introduces convenient notation for the algebra underlying the calculations in this and subsequent sections.

quadratic in $x = (x_1', x_2')'$:

$$\log f(x) = \Phi_0 + \Phi_1 x_1 + \Phi_2 x_2 + \Phi_3 (x_1 \otimes x_1) \tag{23}$$

Then conditional expectations are of the form:

$$
\begin{aligned}
\log E\left[\left(\frac{M_{t+1}}{M_t}\right) f(X_{t+1}) \mid X_t = x\right] &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) f(X_{t+1}) \mid X_t = x\right] \\
&= \Phi_0^* + \Phi_1^* x_1 + \Phi_2^* x_2 + \Phi_3^* (x_1 \otimes x_1) \\
&= \log f^*(x) \tag{24}
\end{aligned}
$$

where the formulas for Φ_i^*, $i = 0, \ldots, 3$ are given in Appendix A. This calculation maps a function f into another function f^* with the same functional form. Our multi-period calculations exploit this link. For instance, repeating these calculations compounds stochastic growth or discounting. Moreover, we may exploit the recursive Markov construction in (24) initiated with $f(x) = 1$ to obtain:

$$\log E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right] = \Phi_{0,t}^* + \Phi_{1,t}^* x_1 + \Phi_{2,t}^* x_2 + \Phi_{3,t}^* (x_1 \otimes x_1)$$

for appropriate choices of $\Phi_{i,t}^*$.

5.1.1   Shock elasticities

To compute shock elasticities given in (8) under the convenient functional form, we construct:

$$\frac{E\left[\left(\frac{M_t}{M_0}\right) W_1 \mid X_0 = x\right]}{E\left[\left(\frac{M_t}{M_0}\right) \mid X_0 = x\right]} = \frac{E\left[\left(\frac{M_1}{M_0}\right) E\left[\left(\frac{M_t}{M_1}\right) \mid X_1\right] W_1 \mid X_0 = x\right]}{E\left[\left(\frac{M_1}{M_0}\right) E\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right]}.$$

Notice that the random variable:

$$L_{1,t} = \frac{\left(\frac{M_1}{M_0}\right) E\left(\frac{M_t}{M_1} \mid X_1\right)}{E\left[\left(\frac{M_1}{M_0}\right)\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right]} \tag{25}$$

has conditional expectation one. Multiplying this positive random variable by W_1 and taking expectations is equivalent to changing the conditional probability distribution and evaluating the conditional expectation of W_1 under this change of measure. Then under the transformed measure, using a complete-the-squares argument we may show that W_1 remains normally distributed with a covariance matrix that is no longer the identity and a mean conditioned on $X_0 = x$ that is affine in x_1. The formulas are given in Appendix B. Thus the shock elasticity function $\varepsilon(x, t)$ can be computed recursively using formulas that are straightforward to implement. We show in Appendix B that the resulting shock elasticity function is also affine in the state x_1.

5.2 Perturbation methods

In macroeconomic models, the equilibrium Markov dynamics (1) is typically ex-ante unknown and needs to be solved for from a set of equilibrium conditions. We now describe a solution method for dynamic general equilibrium models that yields a solution in the form of an approximate law of motion that is a special case of the exponential-quadratic functional form analyzed in Section 5.1. This solution method, based on Holmes (1995) and Lombardo and Uhlig (2014), constructs a perturbation approximation where the first- and second-order terms follow the restricted dynamics (21).

For the purposes of approximation, we consider a family of models parameterized by q and study first- and second-order approximations around this limit system in which $\mathsf{q} = 0$. For each q, we consider the system (equations

$$0 = E\left(g\left[X_{t+1}\left(\mathsf{q}\right), X_t\left(\mathsf{q}\right), X_{t-1}\left(\mathsf{q}\right), \mathsf{q}W_{t+1}, \mathsf{q}W_t, \mathsf{q}\right] \mid \mathcal{F}_t\right). \tag{26}$$

The $\mathsf{q} = 0$ equation system is one without shocks, and more generally small values q will make the shocks less consequential. There are well-known saddle-point stability conditions on the system (26) that lead to a unique equilibrium of the linear approximation (see Blanchard and Kahn (1980) or Sims (2002)) and we assume that these are satisfied. Following Holmes (1995) and Lombardo and Uhlig (2014), we form an approximating system by deducing the dynamic evolution for the pathwise derivatives with respect to q and evaluated at $\mathsf{q} = 0$. Our derivation will be admittedly heuristic as is much of the related literature in macroeconomics.

To build a link to the parametrization in Section 5.1, we feature a second-order expansion:

$$X_t\left(\mathsf{q}\right) \approx X_{0,t} + \mathsf{q}X_{1,t} + \frac{\mathsf{q}^2}{2}X_{2,t}.$$

where $X_{m,t}$ is the m-th order, date t component of the stochastic process. We abstract from the dependence on initial conditions by restricting each component process to be stationary. Our approximating process will similarly be stationary.[13] The expansion leads to laws of motion for the component processes $X_{1,\cdot}$ and $X_{2,\cdot}$. The joint process $(X_{1,\cdot}, X_{2,\cdot})$ will again be Markov, although the dimension of the state vector under the approximate dynamics doubles.

5.2.1 Approximating state vector dynamics

While $X_t\left(\mathsf{q}\right)$ serves as a state vector in the dynamic system (26), the state vector itself depends on the parameter q. Suppose that $\mathcal{F}_t$ is the σ-algebra generated by the infinite history of shocks $\{W_j : j \leq t\}$. For each dynamic system, we presume that the state vector $X_t\left(\mathsf{q}\right)$ is $\mathcal{F}_t$ measurable and that in forecasting future values of the state vector conditioned on $\mathcal{F}_t$ it suffices to condition on X_t. Although $X_t\left(\mathsf{q}\right)$ depends on q, the construction of $\mathcal{F}_t$ does not. We now construct the

---

[13]As argued by Lombardo and Uhlig (2014), this approach is computationally very similar to the pruning approach described by Kim et al. (2008) or Andreasen et al. (2010).

dynamics for each of the component processes. The result will be a recursive system that has the same structure as the triangular system (21).

Define $\bar{x}$ to be the solution to the equation:

$$\bar{x} = \psi(\bar{x}, 0, 0),$$

which gives the fixed point for the deterministic dynamic system. We assume that this fixed point is locally stable. That is $\psi_x(\bar{x}, 0, 0)$ is a matrix with stable eigenvalues, eigenvalues with absolute values that are strictly less than one. Then set

$$X_{0,t} = \bar{x}$$

for all t. This is the zeroth-order contribution to the solution constructed to be time-invariant.

In computing pathwise derivatives, we consider the state vector process viewed as a function of the shock history. Each shock in this history is scaled by the parameter q, which results in a parameterized family of stochastic processes. We compute derivatives with respect to this parameter where the derivatives themselves are stochastic processes. Given the Markov representation of the family of stochastic processes, the derivative processes will also have convenient recursive representations. In what follows we derive these representations.

Using the Markov representation, we compute the derivative of the state vector process with respect to q, which we evaluate at $\mathsf{q} = 0$. This derivative has the recursive representation:

$$X_{1,t+1} = \psi_q + \psi_x X_{1,t} + \psi_w W_{t+1}$$

where ψ_q, ψ_x and ψ_w are the partial derivative matrices:

$$\psi_q \doteq \frac{\partial \psi}{\partial \mathsf{q}}(\bar{x}, 0, 0), \quad \psi_x \doteq \frac{\partial \psi}{\partial x'}(\bar{x}, 0, 0), \quad \psi_w \doteq \frac{\partial \psi}{\partial w'}(\bar{x}, 0, 0).$$

In particular, the term $\psi_w W_{t+1}$ reveals the role of the shock vector in this recursive representation. Recall that we have presumed that $\bar{x}$ has been chosen so that ψ_x has stable eigenvalues. Thus the first derivative evolves as a Gaussian vector autoregression. It can be expressed as an infinite moving average of the history of shocks, which restricts the process to be stationary. The first-order approximation to the original process is:

$$X_t \approx \bar{x} + \mathsf{q} X_{1,t}.$$

In particular, the approximating process on the right-hand side has $\bar{x} + \mathsf{q}(I - \psi_x)^{-1}\psi_q$ as its unconditional mean.

We compute the pathwise second derivative with respect to q recursively by differentiating the recursion for the first derivative. As a consequence, the second derivative has the recursive

representation:

$$X_{2,t+1} = \psi_{qq} + 2\left(\psi_{xq}X_{1,t} + \psi_{wq}W_{t+1}\right) + \qquad\qquad\qquad (27)$$
$$+\psi_x X_{2,t} + \psi_{xx}\left(X_{1,t} \otimes X_{1,t}\right) + 2\psi_{xw}\left(X_{1,t} \otimes W_{t+1}\right) + \psi_{ww}\left(W_{t+1} \otimes W_{t+1}\right)$$

where matrices ψ_{ij} denote the second-order derivatives of ψ evaluated at $(\bar{x}, 0, 0)$ and formed using the construction of the derivative matrices described in Appendix A.2. As noted by Schmitt-Grohé and Uribe (2004), the mixed second-order derivatives ψ_{xq} and ψ_{wq} are often zero using second-order refinements to the familiar log approximation methods.

The second-derivative process $X_{2,\cdot}$ evolves as a stable recursion that feeds back on itself and depends on the first derivative process. We have already argued that the first derivative process $X_{1,t}$ can be constructed as a linear function of the infinite history of the shocks. Since the matrix ψ_x has stable eigenvalues, $X_{2,t}$ can be expressed as a linear-quadratic function of this same shock history. Since there are no feedback effects from $X_{2,t}$ to $X_{1,t+1}$, the joint process $(X_{1,\cdot}, X_{2,\cdot})$ constructed in this manner is necessarily stationary.

The dynamic evolution for $(X_{1,\cdot}, X_{2,\cdot})$ is a special case of the the triangular system (21) given in Section 5.1. When the shock vector W_t is a multivariate standard normal, we can utilize results from Section 5.1 to produce exact formulas for conditional expectations of exponentials of linear-quadratic functions in $(X_{1,t}, X_{2,t})$. We exploit this construction in the subsequent subsection. For details on the derivation of the approximating formulas see Appendix A.

5.3 Approximating the evolution of a stationary increment process

Consider the approximation of a parameterized family of multiplicative processes with increments given by:

$$\log M_{t+1}(\mathsf{q}) - \log M_t(\mathsf{q}) = \kappa[X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}]$$

and an initial condition $\log M_0$. We use the function κ in conjunction with q to parameterize implicitly a family of additive functionals. We approximate the resulting additive functionals by

$$\log M_t \approx \log M_{0,t} + \mathsf{q}\log M_{1,t} + \frac{\mathsf{q}^2}{2}\log M_{2,t}$$

where the processes on the right-hand side have stationary increments.

Following the steps of our approximation of X, the recursive representation of the zeroth-order contribution to Y is

$$\log M_{0,t+1} - \log M_{0,t} = \kappa(\bar{x}, 0, 0) \doteq \bar{\kappa};$$

the first-order contribution is

$$\log M_{1,t+1} - \log M_{1,t} = \kappa_q + \kappa_x X_{1,t} + \kappa_w W_{t+1}$$

where κ_x and κ_w are the respective first derivatives of κ evaluated at $(\bar{x}, 0, 0)$; and the second-order

contribution is

$$
\begin{aligned}
\log M_{2,t+1} - \log M_{2,t} \quad = \quad & \kappa_{qq} + 2\left(\kappa_{xq}X_{1,t} + \kappa_{wq}W_{t+1}\right) + \\
& + \kappa_x X_{2,t} + \kappa_{xx}\left(X_{1,t} \otimes X_{1,t}\right) + 2\kappa_{xw}\left(X_{1,t} \otimes W_{t+1}\right) + \kappa_{ww}\left(W_{t+1} \otimes W_{t+1}\right)
\end{aligned}
$$

where the κ_{ij}'s are the second derivative matrices constructed as in Appendix A.2. The resulting component additive functionals are special cases of the additive functional given in (22) that we introduced in Section 5.1.

5.3.1 Approximating shock elasticities

We could compute corresponding second-order approximations for the elasticities of multiplicative processes. Alternatively, since the approximating processes satisfy the structure given in section 5.1, we have the formulas that we described earlier at our disposal and the supporting software. See Borovička and Hansen (2014) for further discussion.

5.4 Related approaches

There also exist *ad-hoc* approaches which mix orders of approximation for different components of the model or state vector. The aim of these methods is to improve the precision of the approximation along specific dimensions of interest, while retaining tractability in the computation of the derivatives of the function ψ. Justiniano and Primiceri (2008) use a first-order approximations but augment the solution with heteroskedastic innovations. Benigno et al. (2010) study second-order approximations for the endogenous state variables in which exogenous state variables follow a conditionally linear Markov process. Malkhozov and Shamloo (2011) combine a first-order perturbation with heteroskedasticity in the shocks to the exogenous process and corrections for the variance of future shocks. These solution methods are designed to produce nontrivial roles for stochastic volatility in the solution of the model and in the pricing of exposure to risk. The approach of Benigno et al. (2010) or Malkhozov and Shamloo (2011) give alternative ways to construct the functional form used in Section 5.1.

5.5 Recursive utility investors

The recursive utility preference specification of Kreps and Porteus (1978), Epstein and Zin (1989) warrant special consideration. By design, this specification of preferences avoids presuming that investors reduce intertemporal, compound consumption lotteries. Instead investors may care about the intertemporal composition of risk. It is motivated in part by an aim to allow for risk aversion to be altered without changing the elasticity of intertemporal substitution. Anderson et al. (2003), Maenhout (2004) and others extend the literature on risk-sensitive control by Jacobson (1973), Whittle (1990) and others and provide a "concern for robustness" interpretation of the utility recursion. Under this alternative interpretation the decision maker explores the potential misspecification of the transition dynamics as part of the decision-making process. This perspective yields

a substantially different interpretation of the utility recursion. In establishing these connections in the control theory and economics literatures, it is sometimes advantageous to parameterize the utility recursion in a manner that depends explicitly on the parameter q. Borovička and Hansen (2013) and Bhandari et al. (2016) explore the resulting implications for approximations analogous to those studied here. Among other things, they provide a rationale for the first-order adjustments for recursive utility as suggested by Tallarini (2000), and they show novel ways in which higher-order adjustments are more impactful.

6   Continuous-time approximation

Many interesting macroeconomic models specified in continuous time, including those we analyze in Section 7, require the application of numerical solution techniques. In the construction of shock elasticities, the central object of interest is the conditional expectation of M in (19). Consider the more general problem

$$\phi_t(x) \doteq E\left[\left(\frac{M_t}{M_0}\right)\phi_0(X_t) \mid X_0 = x\right]. \tag{28}$$

with a given function ϕ_0. The conditional expectation of M is obtained by setting $\phi_0(x) \equiv 1$.

6.1   An associated partial differential equation

For the purposes of computation, we evaluate ϕ_t recursively. Given $\phi_{t-\Delta t}$ for small Δt, exploiting the time homogeneity of the underlying Markov process and applying the Law of Iterated Expectations gives:

$$\phi_t(x) = E\left[\left(\frac{M_{\Delta t}}{M_0}\right)\phi_{t-\Delta t}(X_{\Delta t}) \mid X_0 = x\right]$$

Itô's lemma applied to the product in the conditional expectation gives the linear, second-order partial differential equation:

$$
\begin{aligned}
\frac{\partial}{\partial t}\phi_t &= \left(\beta + \frac{1}{2}|\alpha(x)|^2\right)\phi_t + \left[\frac{\partial}{\partial x}\phi_t\right]\cdot(\mu + \sigma\alpha) + \\
&\quad + \frac{1}{2}\mathrm{tr}\left[\sigma'\left(\frac{\partial}{\partial x\partial x'}\phi_t\right)\sigma\right]
\end{aligned}
\tag{29}
$$

with terminal condition ϕ_0 where $\mathrm{tr}(\cdot)$ denotes the trace of the matrix argument. Equation (29) is a generalization of the Kolmogorov backward equation for multiplicative processes of the type (17). The resulting partial differential equation can be solved using standard numerical techniques for differential equations.

6.2   Martingale decomposition and a change of measure

To study the long-run implications for pricing, we proposed the extraction of a martingale component from the dynamics of the stochastic discount factors and cash flows by solving the Perron–Frobenius equation (9) for the strictly positive eigenfunction $e(x)$ and the associated eigenvalue η.

In the Markov diffusion setup we localize this problem by computing

$$\lim_{t \to 0} \frac{E\left[M_t e(X_t) \mid X_0 = x\right] - \exp(\eta t) e(x)}{t} = 0.$$

Defining the infinitesimal operator

$$\mathbb{B} f(x) \doteq \left. \frac{d}{dt} E\left[M_t f(X_t) \mid X_0 = x\right] \right|_{t=0}$$

we have

$$\mathbb{B} f = \left(\beta + \frac{1}{2}|\alpha|^2\right) f + (\sigma \alpha + \mu) \cdot \frac{\partial f}{\partial x} + \frac{1}{2} \mathrm{tr} \left(\sigma \sigma' \frac{\partial^2 f}{\partial x \partial x'}\right)$$

and we can write the limiting Perron–Frobenius equation as

$$\mathbb{B} e = \eta e \tag{30}$$

which is a second-order partial differential equation for the function $e(x)$ and a number η. Equation (30) is known as the Sturm–Liouville equation. Notice that it is identical to the partial differential equation (29) when we are looking for an unknown discounted stationary function $\phi_t(x) = \exp(\eta t) e(x)$ with initial condition $\phi_0(x) = e(x)$. As before, there are typically multiple strictly positive solutions to this equation. Hansen and Scheinkman (2009) show that there is at most one such solution that preserves stochastic stability of the state vector X. We implicitly assume that we always choose such a solution.[14]

In line with the discussion from Section 3.4, we can now define the martingale $\widetilde{M}$ as[15]

$$\frac{\widetilde{M}_t}{\widetilde{M}_0} \doteq \exp(-\eta t) \frac{e(X_t)}{e(X_0)} \frac{M_t}{M_0}. \tag{31}$$

Applying Itô's lemma, we find that

$$d \log \widetilde{M}_t = \widetilde{\alpha}(X_t) \cdot dW_t - \frac{1}{2} |\widetilde{\alpha}(X_t)| \, dt$$

with

$$\widetilde{\alpha}(x) = \left[\sigma'(x) \frac{\partial}{\partial x} \log e(x) + \alpha(x)\right].$$

This implies that under the probability measure $\widetilde{P}$, the Brownian motion evolves as

$$dW_t = \widetilde{\alpha}(x) \, dt + d\widetilde{W}_t$$

---

[14]See also Borovička et al. (2015), Qin and Linetsky (2014a), Qin et al. (2016), Walden (2014) or Park (2015) for problems closely related to solving for the eigenvalue-eigenfunction pair (η, e).

[15] We note that the solution obtained using the localized version of the Perron–Frobenius problem may yield a process $\widetilde{M}$ that is only a local martingale. See Hansen and Scheinkman (2009) and Qin and Linetsky (2014b) for details and additional assumptions that assure $\widetilde{M}$ is a martingale. We will assume that such conditions are satisfied in the discussion that follows.

where $\widetilde{W}$ is a Brownian motion under $\widetilde{P}$. It also implies that we can write the dynamics of the state vector under the change of measure as

$$dX_t = [\mu(X_t) + \sigma(X_t)\widetilde{\alpha}(X_t)]\, dt + \sigma(X_t)\, d\widetilde{W}_t.$$

Inverting equation (31), we obtain the analog of the martingale decomposition in discrete time:

$$\frac{M_t}{M_0} = \exp(\eta t)\frac{e(X_0)}{e(X_t)}\frac{\widetilde{M}_t}{\widetilde{M}_0}. \tag{32}$$

To implement the factorization of the multiplicative functional M, we compute the strictly positive eigenfunction $e(x)$ and the associated eigenvalue η by solving the Perron–Frobenius problem (30). Since analytical solutions are often not available, we must rely on numerical methods. Pryce (1993) gives various numerical solution techniques for this problem. Notice that since there are typically infinitely many strictly positive solutions $e(x)$, it is necessary determine which of these solutions is the relevant one.

An alternative approach is to utilize the time-dependent PDE (29) and exploit the fact that η is the principal eigenvalue, i.e., one associated with the most durable component. In that case, one can start with an initial condition $\phi_0(x)$ that serves as a guess for the eigenfunction, and iterate on (29) to solve for $\phi_t(x)$ as $t \to \infty$. For large t, the solution should behave as

$$\phi_t(x) \approx \exp(\eta t)\, e(x)$$

and thus

$$\eta = \left.\frac{\partial}{\partial t}\log\phi_t(x)\right|_{t\to\infty} \approx \left.\frac{1}{\Delta t}\left[\log\phi_{t+\Delta t}(x) - \log\phi_t(x)\right]\right|_{t\to\infty}$$

and since the eigenfunction is only determined up to scale, we can use any proportional rescaling of ϕ_t as $e(x) \approx \exp(-\eta t)\phi_t(x)|_{t\to\infty}$.

6.3   Long-term pricing

We now apply the decomposition (32) in the shock elasticity formula (19) to obtain:

$$\varepsilon(x,t) \doteq \nu(x)\cdot\left[\sigma(x)'\left(\frac{\partial}{\partial x}\log e(x) + \frac{\partial}{\partial x}\log\widetilde{E}\left[\frac{1}{e(X_t)}\mid X_0 = x\right]\right) + \alpha(x)\right].$$

Taking the limit as $t \to \infty$, the conditional expectation in brackets converges to a constant provided that we select a martingale that induces a probability measure under which X is stochastically stable. See Hansen and Scheinkman (2009) and Hansen (2012) for further discussion. Therefore,

$$\lim_{t\to\infty}\varepsilon(x,t) = \nu(x)\cdot\left[\sigma(x)'\frac{\partial}{\partial x}\log e(x) + \alpha(x)\right].$$

6.4 Boundary conditions

The construction of shock elasticity functions requires solving the conditional expectations of M, for instance, by solving the partial differential equation (29). This requires proper specification of the boundary conditions not only in terms of the terminal condition $\phi_0(x)$ but also at the boundaries of the state space for the state vector X_t. The boundary behavior of the diffusion X is a central and often economically important part of the equilibrium, as we will see in the models with financial frictions discussed in Section 7. In those models, the state variable is a univariate diffusion and there are well understood characterizations of the boundary behavior based on the classical Feller boundary classification.[16] The textbook treatment of the boundary conditions for problem (28) typically abstracts from the impact of the multiplicative process M. While a detailed discussion of the boundary characterization is beyond the scope of this chapter, we briefly discuss how the inclusion of M can alter the analysis. In what follows, we utilize the martingale decomposition introduced in Section 3.4 and draw connections to the treatment of boundaries for scalar diffusions.

We represent the conditional expectation (32) using a Kolmogorov equation under the change of measure induced by $\widetilde{M}$. Using the martingale factorization (32) we write (28) as

$$\phi_t(x) \doteq E\left[\exp(\eta t)\frac{e(X_0)}{e(X_t)}\frac{\widetilde{M}_t}{\widetilde{M}_0}\phi_0(X_t) \mid X_0 = x\right].$$

Define

$$\psi_t(x) \doteq \exp(-\eta t)\frac{\phi_t(x)}{e(x)} = \widetilde{E}\left[\frac{\phi_0(X_t)}{e(X_t)} \mid X_0 = x\right] = \widetilde{E}[\psi_0(X_t) \mid X_0 = x] \tag{33}$$

with the initial condition $\psi_0(x) = \phi_0(x)/e(x)$. This converts the boundary condition problem into a standard Kolmogorov backward equation (equation (28) with $M \equiv 1$), albeit under the probability measure $\widetilde{P}$. Under $\widetilde{P}$, the diffusion X satisfies the law of motion

$$dX_t = \widetilde{\mu}(X_t)\,dt + \sigma(X_t)\,d\widetilde{W}_t.$$
$$\widetilde{\mu}(x) = \mu(x) + \sigma(x)\sigma'(x)\frac{\partial}{\partial x}\log e(x) + \sigma(x)\alpha(x)$$

and the associated generator

$$\widetilde{\mathbb{B}}f = \widetilde{\mu} \cdot \frac{\partial f}{\partial x} + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'\frac{\partial^2 f}{\partial x \partial x'}\right)$$

corresponds to the generator of a diffusion with infinitesimal variance $\sigma^2(x)$ and infinitesimal mean $\widetilde{\mu}(x)$ under $\widetilde{P}$.

The boundary characterization under $\widetilde{P}$ and the associated boundary conditions for $\psi_t(x)$ follow from formulas from Section 6.4. The character of the boundary can change under $\widetilde{P}$, although a reflecting boundary remains reflecting to preserve local equivalence of measures P and $\widetilde{P}$. Observe

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[16] See the seminal work by Feller (1952, 1957). Karlin and Taylor (1981), Borodin and Salminen (2002) or Linetsky (2008) offer summarizing treatments.

that equation (33) introduces a relationship between the conditional expectation given by $\phi_t(x)$ and the eigenfunction $e(x)$. For instance, when the boundary point x_b is reflecting, the appropriate boundary condition is[17]

$$\frac{\partial}{\partial x}\psi_t(x)\bigg|_{x=x_b} = 0.$$

When both $\phi_t(x)$ and $e(x)$ are strictly positive at the boundary, this implies that

$$\frac{\partial}{\partial x}\log\phi_t(x)\bigg|_{x=x_b} = \frac{\partial}{\partial x}\log e(x)\bigg|_{x=x_b}$$

equalizing logarithmic slopes of the conditional expectation (28) and the eigenfunction $e(x)$ at the boundary.

7   Models with financial constraints in continuous time

Recently, there has been renewed interest in nonlinear stochastic macroeconomic models with financing restrictions. The literature was initiated by Bernanke and Gertler (1989) and Bernanke et al. (1999), and it has been revived and extended since the advent the financial crisis. Continuous-time models have been featured in He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), Di Tella (2015), Moreira and Savov (2016), Adrian and Boyarchenko (2012), or Klimenko et al. (2016). Differential equation methods give the equilibrium solutions, and the resulting dynamics exhibit quantitatively substantial nonlinearity. The nonlinearity emerges because of financing constraints that bind only in a specific part of the state space.[18]

To preserve tractability, models typically assume a low-dimensional specification of the state space. In this section, we analyze two such models, He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014). Both models utilize frameworks that are judiciously chosen to lead to a scalar endogenous state variable that follows the diffusion

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t. \tag{34}$$

The endogenous state represents the allocation of wealth between households and financial experts, capturing the capitalization of the financial sector relative to the size of the economy. When the capitalization is low, the financial constraint is binding, and asset valuations are more sensitive to aggregate shocks.

Both papers also feature an exogenous process that introduces aggregate risk into their model economies. He and Krishnamurthy (2013) construct an endowment economy with a permanent shock to the aggregate dividend. On the other hand, Brunnermeier and Sannikov (2014) feature endogenous capital accumulation with a a shock to the quality of the capital stock. In this section, we utilize the continuous-time tools developed in Section 6 to study the state-dependence in asset

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[17]This assumes that the so called scale measure is finite at the boundary, see, e.g., Borodin and Salminen (2002).

[18]See Bocola (2016) or Bianchi (2016) for discrete-time models solved using global to account for financing constraint that binds only occasionally.

pricing implications of the two models. We refer the reader to the respective papers for a detailed discussions of the underlying economic environments.

7.1 Stochastic discount factors

Stochastic discount factors and priced cash flows in the models we analyze can be written as special cases of multiplicative functionals introduced in Section 4.2:

$$d \log S_t = \beta\left(X_t\right) dt + \alpha\left(X_t\right) \cdot dW_t. \tag{35}$$

with coefficients $\beta(x)$ and $\alpha(x)$ determined in equilibrium. In an arbitrage-free, complete market environment, there exists a unique stochastic discount factor that represents the prices of the traded securities.

In economies with financial market imperfections and constraints, this ceases to be true. There are two key features that are of interest to us. First, financial markets in these economies are segmented, and different investors can own specific subsets of assets. This implies the existence of alternative stochastic discount factors for individual investors that have to agree only on prices of assets traded between investors. Second, assets are valuable not only for their cash flows but also because their ownership can relax or tighten financing constraints faced by individual investors. Given the potential for these constraints to be binding, asset values include contributions from the shadow prices of these constraints.

7.2 He and Krishnamurthy (2013)

He and Krishnamurthy (2013) construct an economy populated by two types of agents, specialists and households. There are two assets in the economy, a safe asset earning an infinitesimal risk-free rate r_t and a risky asset with return R_t that is a claim on aggregate dividend

$$d \log D_t = \left(g_d - \frac{1}{2}\sigma_d^2\right) dt + \sigma_d dW_t \doteq \bar{\beta}_d dt + \bar{\alpha}_d dW_t. \tag{36}$$

7.2.1 Households and specialists

Households have logarithmic preferences and therefore consume a constant fraction of their wealth, $C_t^h = \rho A_t^h$, where ρ is the time-preference coefficient. A fraction λ of households can only invest into the safe asset, while a fraction $1 - \lambda$ invests a share α_t^h of their wealth through an intermediary managed by the specialists who holds a portfolio with return $d\widetilde{R}_t$. Aggregate wealth of the households therefore evolves as

$$dA_t^h = \left(\ell D_t - \rho A_t^h\right) dt + A_t^h r_t dt + \alpha_t^h\left(1 - \lambda\right) A_t^h \left(d\widetilde{R}_t - r_t dt\right).$$

where ℓD_t is households' income, modeled as a constant share ℓ of the dividend.

Specialists are endowed with CRRA preferences over their consumption stream C_t with risk aversion coefficient γ and trade both assets. Their stochastic discount factor is

$$\frac{S_t}{S_0} = e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{-\gamma}. \tag{37}$$

This stochastic discount factor also prices all assets traded by specialists. The law of motion for their wealth is given by

$$dA_t = -C_t dt + A_t r_t dt + A_t \left(d\widetilde{R}_t - r_t dt \right).$$

The intermediary combines all wealth of the specialists A_t with the households' wealth invested through the intermediary $\alpha_t^h (1 - \lambda) A_t^h$ and invests a share α_t of the combined portfolio into the risky asset. The return on the intermediary portfolio then follows

$$d\widetilde{R}_t = r_t dt + \alpha_t \left(dR_t - r_t dt \right).$$

The risky asset market clears, so that the wealth invested into the risky asset equals the market price of the asset, P_t

$$\alpha_t \left(A_t + \alpha_t^h (1 - \lambda) A_t^h \right) = P_t.$$

7.2.2   Financial friction

The critical financial friction is introduced into the portfolio choice of the household. Motivated by a moral hazard problem, the household is not willing to invest more than a fraction m of the specialists' wealth through the intermediary, which defines the *intermediation constraint*

$$\alpha_t^h (1 - \lambda) A_t^h \leq m A_t. \tag{38}$$

Because of logarithmic preferences, the portfolio choice α_t^h of the household is static. The household is also not allowed to sell short any of the assets, so that it solves

$$\max_{\alpha_t^h \in [0,1]} \alpha_t^h E \left[d\widetilde{R}_t - r_t dt \mid \mathcal{F}_t \right] - \frac{1}{2} \left(\alpha_t^h \right)^2 Var \left[d\widetilde{R}_t - r_t dt \mid \mathcal{F}_t \right]$$

subject to the intermediation constraint (38).

The parameter m determines the tightness of the intermediation constraint. This constraint will be endogenously binding when the wealth of the specialists becomes sufficiently low relative to the wealth of the household. In that case, risk sharing partially breaks down and the specialists will have to absorb a large share of the risky asset in their portfolio. As an equilibrium outcome, risk premia increase and the wealth of the specialists becomes more volatile, which in turn induces larger fluctuations of the right-hand side of the constraint (38). Without the intermediation constraint, the model reduces to an endowment economy populated by agents solving a risk-sharing problem with portfolio constraints.

7.2.3 Equilibrium dynamics

The equilibrium in this model is conveniently characterized using the wealth share of the specialists, $X_t \doteq A_t/P_t \in (0,1)$, that will play the role of the single state variable with endogenously determined dynamics (34) where the coefficients $\mu(x)$ and $\sigma(x)$ are given by the relative wealth accumulation rates of households and specialists, and the equilibrium price of the claim on the risky cash flow. He and Krishnamurthy (2013) show that both boundaries $\{0,1\}$ are entrance boundaries.

Given the homogeneity in the model, we can write the consumption of the specialists as

$$
\begin{aligned}
C_t &= D_t(1+\ell) - C_t^h = D_t\left[(1+\ell) - \frac{C_t^h}{A_t^h}\frac{A_t^h}{P_t}\frac{P_t}{D_t}\right]\\
&= D_t\left[(1+\ell) - \rho(1-X_t)\pi(X_t)\right]
\end{aligned}
$$

where $\pi(x)$ is the price-dividend ratio for the claim on the dividend stream. The price-dividend ratio is determined endogenously as part of the solution to a set of differential equations. Given a solution for the price-dividend ratio $\pi(x)$, we construct the stochastic discount factor (37).

The top row of Figure 1 shows the drift and volatility coefficients of the state variable process X, and the associated stationary density. When the specialists' wealth share X_t is low (below $x^* = 0.091$), the intermediation constraint binds. As $X_t \to 0$, the intermediation capacity of the specialists decreases, which increases the expected return on the risky asset, thereby increasing the rate of wealth accumulation of the specialists. On the other hand, when $X_t \to 1$, the economy is unconstrained, risk premia are low, and situation reverses. The drift coefficient $\mu(x)$ in the top left panel reflects these effects.

In the moment when the constraints start binding (to the left of the point $x^* = 0.091$), volatility $\sigma(x)$ of the experts' wealth share starts rising. Ultimately, this volatility has to decline to zero as $X_t \to 0$ to prevent the experts' wealth share from hitting the zero boundary with a positive probability, but the volatility of experts' wealth *level* keeps rising as we approach the boundary.

7.2.4 Stochastic discount factor and cash flows

Aggregate dividend D_t in (36) follows a geometric Brownian motion with drift. This directly implies a constant shock-exposure elasticity

$$
\varepsilon_d(x,t) = \sigma_d.
$$

Time-variation in expected returns on the claim on the aggregate dividend thus must come solely from the time-variation in prices of risk. In particular, the consumption process of specialists is:

$$
\frac{C_t}{C_0} = \left(\frac{D_t}{D_0}\right)\left[\frac{(1+\ell) - \rho(1-X_t)\pi(X_t)}{(1+\ell) - \rho(1-X_0)\pi(X_0)}\right]. \tag{39}
$$

Notice that the consumption of specialists has the same long-term stochastic growth as the aggregate dividend process. Since the dividend process D is a geometric Brownian motion, we immediately
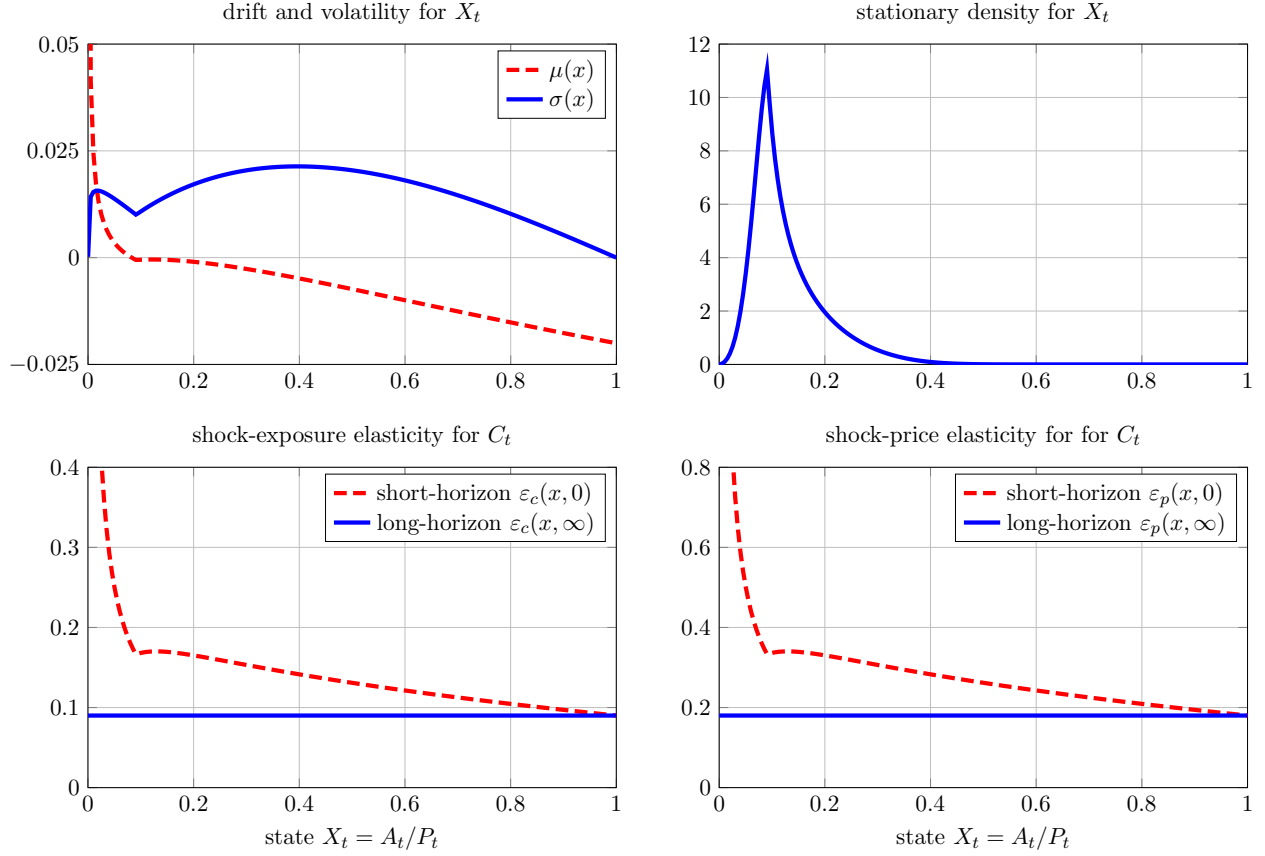
Figure 1: Dynamics of the experts' wealth share $X_t = A_t/P_t$ (horizontal axis), shock-exposure and shock-price elasticities for the He and Krishnamurthy (2013) model. Top left panel shows the drift and volatility coefficients for the evolution of X_t, while top right panel the stationary density for X_t. Panels in the bottom row show the short- and long-horizon shock elasticity for the experts' consumption process C_t. The intermediation constraint (38) binds in the interval $X_t \in (0, 0.091)$, and $x^* = 0.091$ corresponds to the 35.3% quantile of the stationary distribution of X_t.

obtain the martingale factorization of C with

$$
\begin{aligned}
e_c(x) &= [(1 + \ell) - \rho(1 - x)\pi(x)]^{-1} \\
\eta_c &= g_d \\
\widetilde{C}_t &= \exp(-\eta_c t)D_t
\end{aligned}
$$

where $\widetilde{C}$ is the martingale component of C. Analogously, the stochastic discount factor of the specialists (37) is decomposed as

$$
\begin{aligned}
e_s(x) &= [(1 + \ell) - \rho(1 - x)\pi(x)]^{\gamma} \\
\eta_s &= -\rho - \gamma g_d + \frac{1}{2}\sigma_d^2\gamma(\gamma + 1) \\
\widetilde{S}_t &= \exp[(-\eta_s - \rho)t](D_t)^{-\gamma}
\end{aligned}
$$

Figure 2: Shock-exposure and shock-price elasticities for the He and Krishnamurthy (2013) model. Individual lines correspond to alternative choices of the current state, the experts' wealth share $X_0 = x$. The blue solid line represents the state in which the intermediation constraint (38) starts binding ($x = 0.091$, corresponding to the 35.3% quantile of the stationary distribution of X_t. The red dashed line corresponds to the 5% quantile of the st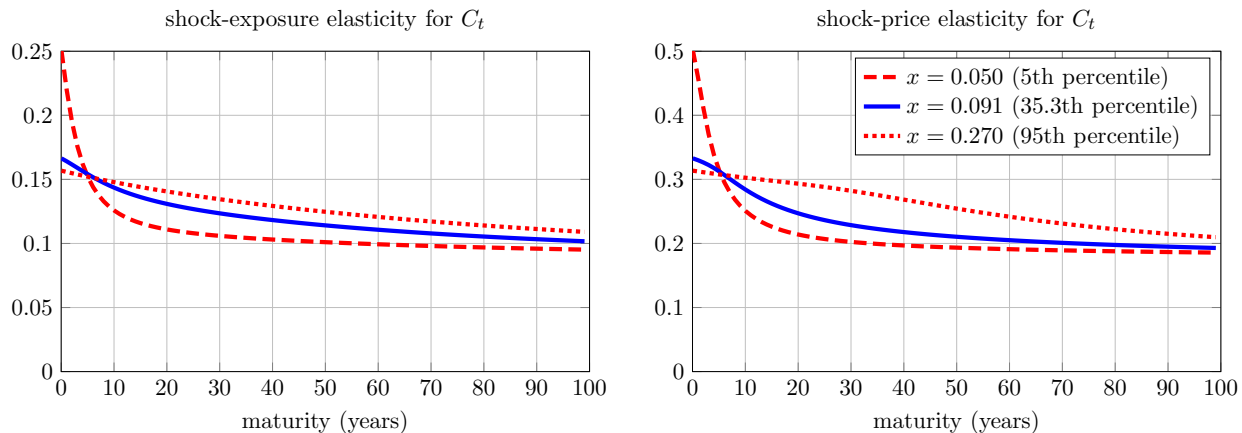ationary distribution of X_t (intermediation constraint tightly binding) while the red dotted line corresponds to the 95% quantile.

where $\widetilde{S}$ is the martingale component.

These factorization results indicate a simple form for the long-horizon limits of the shock elasticities. The consumption and dividend processes share the same martingale component, and thus, assuming $\nu(x) = 1$, their shock-exposure elasticities imply

$$\lim_{t \to \infty} \varepsilon_c(x, t) = \lim_{t \to \infty} \varepsilon_d(x, t) = \sigma_d.$$

Similarly, the shock-price elasticities for the two cash-flow processes have the common long-horizon limit

$$\lim_{t \to \infty} \varepsilon_p(x, t) = \gamma \sigma_d.$$

As we have just verified, the intermediation constraint does not have any impact on prices of long-horizon cash flows. Long-horizon shock elasticities behave as in an economy populated only by unconstrained specialists with risk aversion γ who consume the whole dividend stream D_t. The intermediation constraint only affects the stationary part $e_s(x)$ of the stochastic discount factor.[19] As a consequence, long-term risk adjustments in this model are the same as those implied by a model with power utility function and consumption equal to dividends. The financing constraint induces deviations in short-term risk prices, which we now characterize.
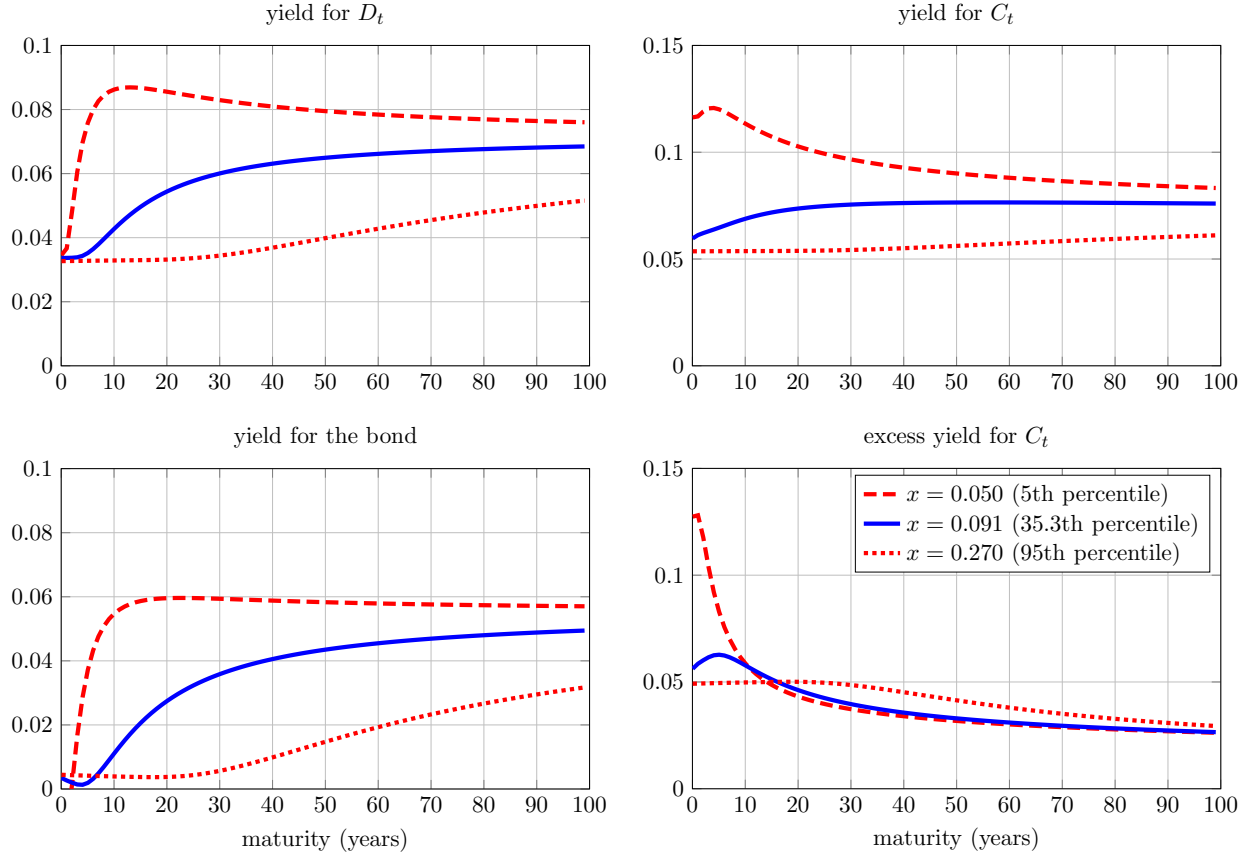
Figure 3: Yields and excess yields for the He and Krishnamurthy (2013) model. Parametrization and description as in Figure 2.

7.2.5 Shock elasticities and term structure of yields

The blue solid lines in the bottom row of Figure 1 represent the long-horizon shock-exposure and shock-price elasticities. These results are contrasted with the infinitesimal shock-exposure and shock-price elasticities, depicted with red dashed lines, that are equal to the volatility coefficients $\alpha_c(x)$ and $\alpha_s(x)$ in the differential representation (35) for the experts' consumption process (39) and stochastic discount factor process (37), respectively.

Figure 2 depicts these shock elasticities evaluated at three different points in the state space. These elasticities were computed numerically.[20] A remarkable feature of the model is the following. The short-horizon consumption cash flows are more exposed to risk as revealed by a larger shock-price elasticity in the constrained region of the state space ($x = 0.05$). This finding is reversed for long-horizon cash flows, showing that the term structure of risk prices is much more strongly downward sloping for low values of the state variable. Since the state variable responds positively

---

[19]Without the intermediation constraint and the debt constraint ($\lambda = 0$), the economy reduces to a complete-market risk-sharing problem between households and specialists and will converge in the long-run to a homogeneous-agent economy populated only by households when $\gamma > 1$.

[20]We solved equation (29) for $M = C$ and $M = SC$, with $\phi_0(x) = 1$ using an implicit finite difference scheme. We used the solution for $\pi(x)$ constructed using the code from He and Krishnamurthy (2013).

to shocks, low realizations of the state variable are the consequence of adverse shocks in the past.

Figure 3 explores the implications for yields on dividends and experts' consumptions for alternative payoff horizons computed as logarithms of expected returns to the respective payoffs. While the yields on dividends and experts' consumption are initially increasing in maturity, this is all the more so when x is low. The yields are monotone over all horizons except when x is low, in which case the yields eventually decline a bit. The same effect is even more pronounced for the risk-free yield curve except the eventual decline is even more slight. Excess yields are therefore downward sloping for the experts' consumption process, and are lower for longer maturities for low values of x in contrast to high values.[21]

7.3 Brunnermeier and Sannikov (2014)

Brunnermeier and Sannikov (2014) construct a model with endogenous capital accumulation, populated by two types of agents, households and experts. The experts have access to a more productive technology for output and new capital than the households. The state variable of interest is the wealth share of experts, defined as

$$X_t = \frac{N_t}{Q_t K_t}$$

where N_t is the net worth of the experts, and $Q_t K_t$ is the market value of capital. The equilibrium stock of capital evolves as

$$d \log K_t = \beta_k (X_t) \, dt + \bar{\alpha}_k dW_t$$

where the rate of accumulation of aggregate capital, $\beta_k (X_t)$, is determined by the wealth share of experts along with a standard local lognormal adjustment. The shock dW_t alters the quality of the capital stock.

7.3.1 Households and experts

In the baseline model, both households and experts have linear preferences and differ in their time-preference coefficients, r and ρ, respectively, assuming that $\rho > r$. In particular, the preferences for experts are given by

$$E \left[\int_0^\infty e^{-\rho t} dCu_t \mid \mathcal{F}_0 \right]$$

where Cu_t is cumulative consumption and as such is restricted to be a non-decreasing process. In contrast, the cumulative consumption of the household can have negative increments. The linearity in their preferences implies a constant equilibrium rate of interest r.

---

[21]For empirical evidence and modeling of the downward sloping term structure of risky yields see van Binsbergen et al. (2012), van Binsbergen et al. (2013), Ai et al. (2013), Belo et al. (2015), Hasler and Marfè (2015), Lopez et al. (2015), or van Binsbergen and Koijen (2016).

7.3.2 Financial friction

In the model, experts are better at managing the capital stock, making it more productive. This creates a natural tendency to move the capital from the hands of the households to the hands of the experts, who in turn issue financial claims on this capital to the households. Absent any financial frictions, the experts would instantly consume the total value of their own net worth (given their higher impatience and linear utility), and accept households' capital under management by issuing equity claims.

Brunnermeier and Sannikov (2014) assume that experts cannot issue any equity and have to finance all capital purchases using risk-free borrowing. This naturally creates a leveraged portfolio on the side of the experts. When the wealth share of experts X_t decreases, they can intermediate households' capital only by increasing their leverage, and the price of capital $Q(X_t)$ has to fall in order to generate a sufficiently high expected return on capital for the experts to hold this leveraged portfolio.

7.3.3 Equilibrium dynamics

In equilibrium, the expected return on capital has to balance the hedging demand on the side of the experts with the supply of capital from households. Experts' hedging motive (limited willingness to hold a leveraged portfolio) arises from the fact that a leveraged portfolio generates a low return after an adverse realization of the shock dW_t which, at the same time, decreases X_t and therefore increases the future expected return on capital.

On the other hand, when the wealth share of experts X_t increases, the price of capital $Q(X_t)$ increases, and the expected return falls. Define the marginal value of experts' wealth $\Theta_t = \theta(X_t)$ through

$$\Theta_t N_t = E\left[\int_t^\infty e^{-\rho(s-t)} d\mathcal{C}_s \mid \mathcal{F}_t\right]$$

where $d\mathcal{C}$ is the cumulative consumption process of the experts. Linearity of preferences implies that experts' consumption is zero as long as $\Theta_t > 1$. As X_t increases, it reaches an endogenously determined threshold $\bar{x}$ for which $\theta(\bar{x}) = 1$. At this point, the marginal utility of wealth equals the marginal utility of consumption, and experts consume out of their wealth. Consequently, the equilibrium dynamics for the wealth share of experts is given by

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t - X_t d\zeta_t.$$

where $\mu(x)$ and $\sigma(x)$ are endogenously determined coefficients that depend on relative rates of wealth accumulation of experts and households, and the consumption rate of experts $d\zeta_t \doteq d\mathcal{C}_t/N_t > 0$ only if $X_t = \bar{x}$. Formally, the right boundary for the stochastic process X_t behaves as a reflecting boundary. See Brunnermeier and Sannikov (2014) for the construction of μ and σ.

7.3.4 Stochastic discount factor and cash flows

We now turn to the study of asset pricing implications in the model. To construct the shock elasticities, we construct the coefficients $\beta(x)$ and $\alpha(x)$ for the evolution of the stochastic discount factor and priced cash flows modeled as multiplicative functionals (35).

The marginal utility of wealth implies the following stochastic discount factor of the experts:

$$\frac{S_t}{S_0} = \exp(-\rho t)\frac{\theta\left(X_t\right)}{\theta\left(X_0\right)}.$$

The coefficients $\beta_s(x)$ and $\alpha_s(x)$ in the equation for the evolution of the stochastic discount factor functional can be constructed by applying Ito's lemma to this expression taking account of the functional dependence given by $\theta(x)$ and the evolution of X. Observe that this stochastic discount factor does not contain a martingale component. Nevertheless, since the equilibrium local risk-free interest rate is r,

$$\exp(rt)\frac{S_t}{S_0} = \exp\left[(r-\rho)t\right]\frac{\theta\left(X_t\right)}{\theta\left(X_0\right)},$$

must be a positive local martingale. As such, its expectation conditioned on date t information could decline in t implying that long-term interest rates could be higher and in fact converge to ρ. More generally, from the standpoint of valuation, the fat right tail of the process $\theta(X_t)$ could have important consequences for valuation even in the absence of a martingale component for the stochastic discount factor process.

As a priced cash-flow, we consider the aggregate consumption flow process C^a given by

$$C_t^a = \left[a_e\psi\left(X_t\right) + a_h\left[1 - \psi\left(X_t\right)\right] - \iota\left(X_t\right)\right] K_t \tag{40}$$

where $\iota(x)$ is the aggregate investment rate, $\psi(x)$ is the fraction of the capital stock owned by the experts, and $a_e > a_h$ are the output productivities of the experts and households, respectively. Thus C_t^a is equal to aggregate output net of aggregate investment. Aggregate consumption is therefore given as a stationary fraction of aggregate capital. Thus aggregate consumption flow and capital stock processes share a common martingale component.[22]

7.3.5 Shock elasticities and term structure of yields

The top left panel in Figure 4 depicts the drift and volatility coefficients for the state variable X_t. At the right boundary $\bar{x}$, the experts accumulated a sufficiently large share of capital and start consuming. Given their risk neutrality, the boundary behaves as a reflecting boundary. At the left boundary, the situation is notably different. Experts' ability to intermediate capital is limited by their own net worth, and hence their portfolio choice corresponds to an effectively risk averse

---

[22]Brunnermeier and Sannikov (2014) also consider an extension where experts and households are endowed with logarithmic utilities. In that case consumption of both households and experts are given as constant fractions of their respective net worth, and the stochastic discount factor of the experts inherits the martingale component from the reciprocal of the aggregate capital process.
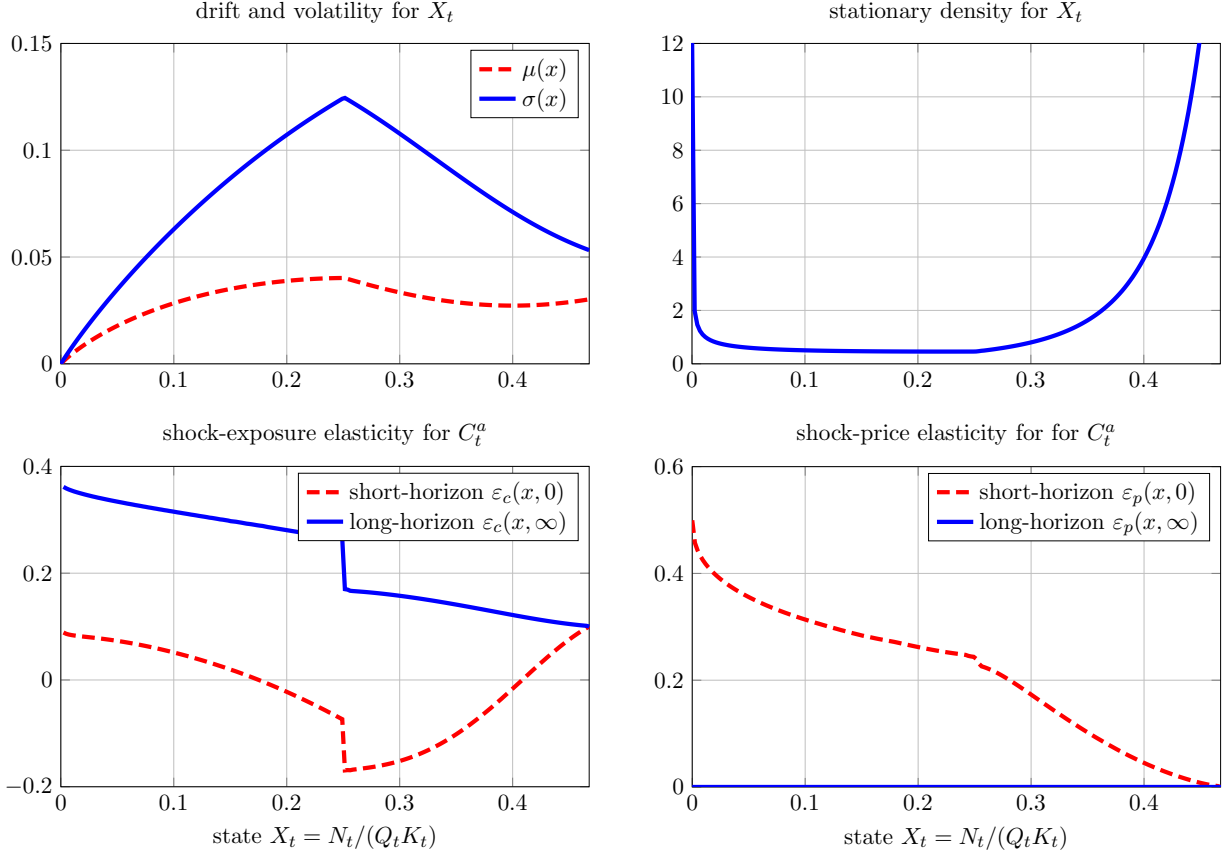
Figure 4: Dynamics of the experts' wealth share $X_t = N_t/(Q_t K_t)$ (horizontal axis), shock-exposure and shock-price elasticities for the Brunnermeier and Sannikov (2014) model. Top left panel shows the drift and volatility coefficients for the evolution of X_t, while top right panel the stationary density for X_t. Panels in the bottom row show the short- and long-horizon shock elasticity for the aggregate consumption process C^a. The intermediation constraint binds in the interval $X_t \in (0, 0.25)$, and $x^* = 0.25$ corresponds to the 15% quantile of the stationary distribution of X.

agent. The left boundary is natural and nonattracting.

The existence of a stationary distribution, depicted in the second panel of Figure 4, arises from a combination of two forces. Experts are more impatient, so whenever they accumulate a sufficient share of capital, they start consuming, which prevents them from taking over the whole economy. On the other hand, when their wealth share falls, their intermediation ability becomes scarce, the expected return on capital rises, and they use their superior investment technology to accumulate wealth at a faster rate than households.

The stationary density has peaks at each of the two boundaries. The positive drift coefficient $\mu(x)$ implies that there is a natural pull toward the right boundary, creating the peak in the density there. However, whenever a sequence of shocks brings the economy close to the left boundary, solvency constraints imply that it takes time for experts to accumulate wealth again, and the economy spends a long period time in that part of the state space. Economically, most times are 'good' times when intermediation is fully operational, with rare periods of protracted 'financial

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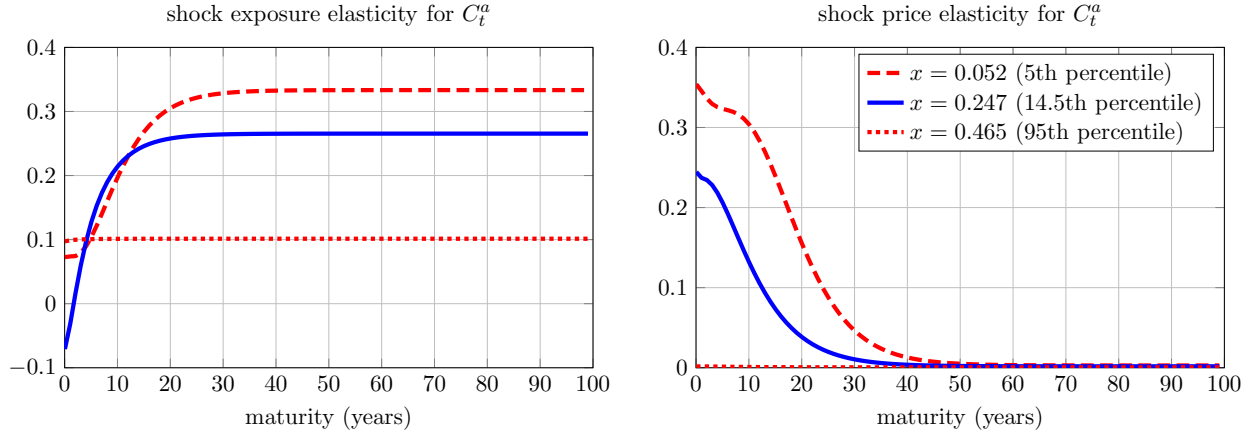
Figure 5: Shock exposure and shock price elasticities for the Brunnermeier and Sannikov (2014) model. Individual lines correspond to alternative choices of the current state, the experts' wealth share $X_0 = x$. The blue solid line represents the state in which the intermediation constraint starts binding ($x = 0.247$), corresponding to the 14.5% quantile of the stationary distribution of X_t. The red dashed line corresponds to the 5% quantile of the stationary distribution of X_t (intermediation constraint tightly binding) while the red dotted line corresponds to the 95% quantile.
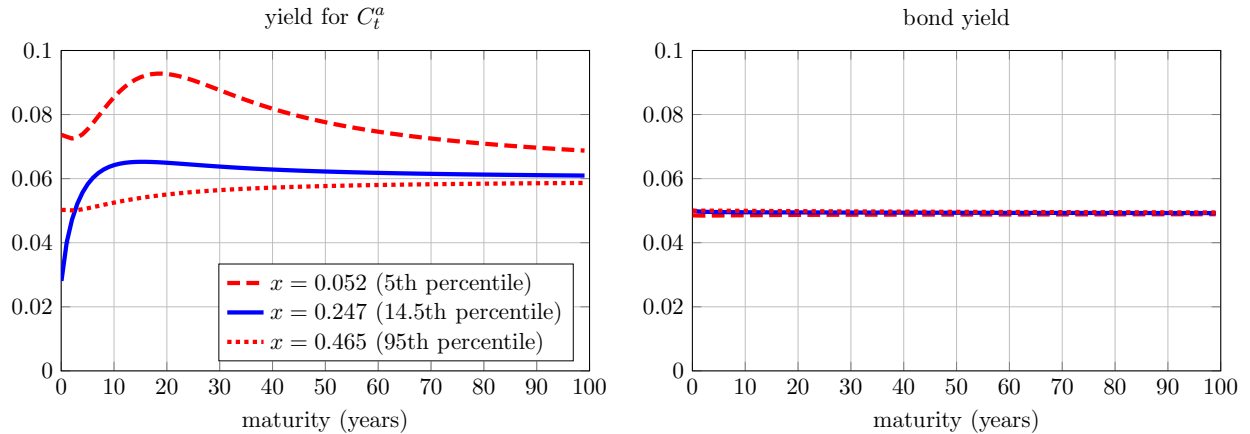
Figure 6: Yields and excess yields for the Brunnermeier and Sannikov (2014) model. Parametrization and description as in Figure 5.

crises'.

The bottom row of Figure 4 plots the shock elasticities for the aggregate consumption process (40). Observe that the short-horizon exposure elasticity is negative in a part of the state space, making aggregate consumption countercyclical there. The long-horizon elasticities are noticeably higher, and particularly high when the intermediation constraint binds. The discontinuity at $X_t = x^*$ is caused by the change in consumption behavior in the moment when the intermediation constraint starts binding.

Given that the stochastic discount factor has no martingale component, the long-horizon shock-price elasticity is zero. On the other hand, the short-horizon price of risk varies strongly with the

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wealth share of the experts. This state dependence is also confirmed in Figure 5 which plots the shock elasticity functions for selected points in the state space. Shock-exposure elasticities for the aggregate consumption process $\{C_t^a : t \geq 0\}$ increase with maturity, while the shock-price elasticities vanish as $t \to \infty$. Notice that there is a sign reversal in the exposure elasticities for aggregate consumption. The shock-exposure elasticities are initially negative but eventually become positive in the middle part of the state space, mirroring the bottom left panel of Figure 4. This pattern emerges because of the equilibrium investment responses over short horizons lead to more substantial longer-term consumption responses in the constrained states. Nevertheless, the shock-price elasticities are positive for all horizons and states that we consider.

Finally, Figure 6 plots the yields on risk-free bonds and claims on horizon-specific cash flows from aggregate consumption. In line with the non-monotonicity of the shock-exposure elasticities across states in Figure 4, the short-maturity yields are also non-monotonic, being lowest, and in fact lower than the risk-free rate, in the center of the distribution of the state X_t.

8   Directions for further research

In this chapter, we developed dynamic value decompositions (DVD's) for the study of intertemporal asset pricing implications of dynamic equilibrium models. We constructed *shock elasticities* as building blocks for these decompositions. The DVD methods are distinct but potentially complementary to the familiar Campbell and Shiller (1988) decomposition. Campbell and Shiller use linear VAR methods to quantify the impact of (discounted) "cash flow shocks" and "expected return shocks" on price-dividend ratios. In general these shocks are correlated and are themselves combinations of shocks that are fundamental to structural models of the macroeconomy. Our aim is to explore pricing implications of models in which alternative macroeconomic shocks are identified and their impact quantified. We replaced linear approximation with local sensitivity analysis, and we characterized how cash flows are exposed to alternative macroeconomic shocks and what the corresponding price adjustments are for these exposures. We showed that shock elasticities are mathematically and economically related to impulse response functions. The shock elasticities represent sensitivities of expected cash flows to alternative macroeconomic shocks and the associated market implied compensations when looking across differing investment horizons.

We apply these DVD methods to a class of dynamic equilibrium models that feature financial frictions and segmented markets. The methods uncover the ways financial frictions contribute to pricing of alternative cash flows and to the shape of the term structure of macroeconomic risk prices.

There are two extensions of our analysis that require further investigation. First risk prices are only well-defined relative to an underlying probability distribution. In this chapter we have not discussed the consequences for pricing when investors inside our models use different probability measures than the data-generating measure presumed by an econometrician. Typically, researchers invoke an assumption of rational expectations to connect investor perceptions with the data generation. More generally, models of investors that allow for subjective beliefs, learning,

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ambiguity aversion or concerns about model misspecification alter how we interpret market based compensations for exposure to macroeconomic fluctuations. For instance, see Hansen (2014) for further discussion. Incorporating potential belief distortions into the analysis should be a valuable extension of these methods.

Second, we left aside empirical and econometric aspects of the identification of shocks and measurement of risk premia. The empirical finance literature has made considerable progress in the characterization and measurement of the term structure of risk premia in various asset markets. The challenge for model building is to connect these empirical facts to specific sources of macroeconomic risks and financial market frictions of model economies. Our methodology suggests a way to make these connections, but further investigation is required.

Finally, we refrained from the discussion of implications for policy analysis. Financial frictions create economic externalities that can potentially be rectified by suitable policy actions. Since asset prices enter these financial constraints, understanding their behavior is an important ingredient to meaningful policy design. Forward looking asset prices provide both a source of information about private sector beliefs and an input into the regulatory challenges faced in the conduct of policy. Our methods can help to uncover asset pricing implications for alternative potential policies.

Appendix

A Exponential-quadratic framework

Let $X = (X_1', X_2')'$ be a $2n \times 1$ vector of states, $W \sim N(0, I)$ a $k \times 1$ vector of independent Gaussian shocks, and $\mathcal{F}_t$ the filtration generated by $(X_0, W_1, \ldots, W_t)$. In this appendix, we show that given the law of motion from equation (21)

$$
\begin{aligned}
X_{1,t+1} &= \Theta_{10} + \Theta_{11} X_{1,t} + \Lambda_{10} W_{t+1} \\
X_{2,t+1} &= \Theta_{20} + \Theta_{21} X_{1,t} + \Theta_{22} X_{2,t} + \Theta_{23} \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Lambda_{20} W_{t+1} + \Lambda_{21} \left(X_{1,t} \otimes W_{t+1} \right) + \Lambda_{22} \left(W_{t+1} \otimes W_{t+1} \right)
\end{aligned}
\tag{41}
$$

and a multiplicative functional $M_t = \exp\left(Y_t \right)$ whose additive increment is given in equation (22):

$$
\begin{aligned}
Y_{t+1} - Y_t &= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3 \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Psi_0 W_{t+1} + \Psi_1 \left(X_{1,t} \otimes W_{1,t+1} \right) + \Psi_2 \left(W_{t+1} \otimes W_{t+1} \right),
\end{aligned}
\tag{42}
$$

we can write the conditional expectation of M as

$$
\log E\left[M_t \mid \mathcal{F}_0 \right] = \left(\bar{\Gamma}_0 \right)_t + \left(\bar{\Gamma}_1 \right)_t X_{1,0} + \left(\bar{\Gamma}_2 \right)_t X_{2,0} + \left(\bar{\Gamma}_3 \right)_t \left(X_{1,0} \otimes X_{1,0} \right)
\tag{43}
$$

where $\left(\bar{\Gamma}_i \right)_t$ are constant coefficients to be determined.

The dynamics given by (41)–(42) embed the perturbation approximation constructed in Section 5.2 as a special case. The Θ and Λ matrices needed to map the perturbed model into the above structure are constructed from the first and second derivatives of the function $\psi(x, w, \mathsf{q})$ that captures the law of motion of the model, evaluated at $(\bar{x}, 0, 0)$:

$$
\begin{aligned}
\Theta_{10} &= \psi_q & \Theta_{11} &= \psi_x & \Lambda_{10} &= \psi_w \\
\Theta_{20} &= \psi_{qq} & \Theta_{21} &= 2\psi_{xq} & \Theta_{22} &= \psi_x & \Theta_{23} &= \psi_{xx} \\
\Lambda_{20} &= 2\psi_{wq} & \Lambda_{21} &= 2\psi_{xw} & \Lambda_{22} &= \psi_{ww}
\end{aligned}
$$

where the notation for the derivatives is defined in Appendix A.2.

A.1 Definitions

To simplify work with Kronecker products, we define two operators vec and $\mathrm{mat}_{m,n}$. For an $m \times n$ matrix H, $\mathrm{vec}\left(H \right)$ produces a column vector of length mn created by stacking the columns of H:

$$
h_{(j-1)m+i} = \left[\mathrm{vec}(\mathrm{H}) \right]_{(j-1)m+i} = H_{ij}.
$$

For a vector (column or row) h of length mn, $\mathrm{mat}_{m,n}\left(h \right)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$
H_{ij} = \left[\mathrm{mat}_{m,n}(h) \right]_{ij} = h_{(j-1)m+i}.
$$

We drop the m, n subindex if the dimensions of the resulting matrix are obvious from the context.

For a square matrix A, define the sym operator as

$$\text{sym}(A) = \frac{1}{2}(A + A').$$

Apart from the standard operations with Kronecker products, notice that the following is true. For a row vector $H_{1 \times nk}$ and column vectors $X_{n \times 1}$ and $W_{n \times 1}$

$$H(X \otimes W) = X' \left[\text{mat}_{k,n}(H)\right]' W$$

and for a matrix $A_{n \times k}$, we have

$$X'AW = (\text{vec}A')' (X \otimes W). \tag{44}$$

Also, for $A_{n \times n}$, $X_{n \times 1}$, $K_{k \times 1}$, we have

$$
\begin{aligned}
(AX) \otimes K &= (A \otimes K) X \\
K \otimes (AX) &= (K \otimes A) X.
\end{aligned}
$$

Finally, for column vectors $X_{n \times 1}$ and $W_{k \times 1}$,

$$(AX) \otimes (BW) = (A \otimes B)(X \otimes W)$$

and

$$(BW) \otimes (AX) = \left[B \otimes A_{\bullet j}\right]_{j=1}^{n} (X \otimes W)$$

where

$$\left[B \otimes A_{\bullet j}\right]_{j=1}^{n} = \left[B \otimes A_{\bullet 1} \quad B \otimes A_{\bullet 2} \quad \ldots \quad B \otimes A_{\bullet n}\right].$$

A.2 Concise notation for derivatives

Consider a vector function $f(x, w)$ where x and w are column vectors of length m and n, respectively. The first-derivative matrix f_i where $i = x, w$ is constructed as follows. The k-th row $[f_i]_{k\bullet}$ corresponds to the derivative of the k-th component of f

$$\left[f_i(x, w)\right]_{k\bullet} = \frac{\partial f^{(k)}}{\partial i'}(x, w).$$

Similarly, the second-derivative matrix is the matrix of vectorized and stacked Hessians of individual components with k-th row

$$\left[f_{ij}(x, w)\right]_{k\bullet} = \left(\text{vec}\frac{\partial^2 f^{(k)}}{\partial j \partial i'}(x, w)\right)'.$$

It follows from formula (44) that, for example,

$$x'\left(\frac{\partial^2 f^{(k)}}{\partial x \partial w'}(x, w)\right)w = \left(\text{vec}\frac{\partial^2 f^{(k)}}{\partial w \partial x'}(x, w)\right)'(x \otimes w) = \left[f_{xw}(x, w)\right]_{k\bullet}(x \otimes w).$$

A.3 Conditional expectations

Notice that a complete-the squares argument implies that, for a $1 \times k$ vector A, a $1 \times k^2$ vector B, and a scalar function $f(w)$,

$$E\left[\exp\left(B\left(W_{t+1} \otimes W_{t+1}\right) + AW_{t+1}\right) f\left(W_{t+1}\right) \mid \mathcal{F}_t\right] = \tag{45}$$
$$= E\left[\exp\left(\frac{1}{2}W'_{t+1}\left(\text{mat}_{k,k}\left(2B\right)\right) W_{t+1} + AW_{t+1}\right) f\left(W_{t+1}\right) \mid \mathcal{F}_t\right]$$
$$= \left|I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right|^{-1/2} \exp\left(\frac{1}{2}A\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1} A'\right) \tilde{E}\left[f\left(W_{t+1}\right) \mid \mathcal{F}_t\right]$$

where $\tilde{\cdot}$ is a measure under which

$$W_{t+1} \sim N\left(\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1} A', \left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2B\right)\right]\right)^{-1}\right).$$

We start by utilizing formula (45) to compute

$$
\begin{aligned}
\bar{Y}\left(X_t\right) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) + \\
&\quad + \log E\left[\exp\left(\left[\Psi_0 + X'_{1t}\left[\text{mat}_{k,n}\left(\Psi_1\right)\right]'\right] W_{t+1} + \frac{1}{2}W'_{t+1}\left[\text{mat}_{k,k}\left(\Psi_2\right)\right] W_{t+1}\right) \mid \mathcal{F}_t\right] \\
&= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) - \\
&\quad - \frac{1}{2}\log\left|I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right| + \frac{1}{2}\mu'\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\mu
\end{aligned}
$$

with μ defined as

$$\mu = \Psi'_0 + \left[\text{mat}_{k,n}\left(\Psi_1\right)\right] X_{1,t}.$$

Reorganizing terms, we obtain

$$\bar{Y}\left(X_t\right) = \bar{\Gamma}_0 + \bar{\Gamma}_1 X_{1,t} + \bar{\Gamma}_2 X_{2,t} + \bar{\Gamma}_3\left(X_{1,t} \otimes X_{1,t}\right) \tag{46}$$

where

$$
\begin{aligned}
\bar{\Gamma}_0 &= \Gamma_0 - \frac{1}{2}\log\left|I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right| + \frac{1}{2}\Psi_0\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\Psi'_0 \\
\bar{\Gamma}_1 &= \Gamma_1 + \Psi_0\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\text{mat}_{k,n}\left(\Psi_1\right)\right] \\
\bar{\Gamma}_2 &= \Gamma_2 \\
\bar{\Gamma}_3 &= \Gamma_3 + \frac{1}{2}\text{vec}\left[\left[\text{mat}_{k,n}\left(\Psi_1\right)\right]'\left(I_k - \text{sym}\left[\text{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\text{mat}_{k,n}\left(\Psi_1\right)\right]\right]'.
\end{aligned}
\tag{47}
$$

For the set of parameters $\mathcal{P} = \left(\Gamma_0, \ldots, \Gamma_3, \Psi_0, \ldots, \Psi_2\right)$, equations (47) define a mapping

$$\bar{\mathcal{P}} = \bar{\mathcal{E}}\left(\mathcal{P}\right),$$

with all $\bar{\Psi}_j = 0$. We now substitute the law of motion for X_1 and X_2 to produce $\bar{Y}\left(X_t\right) = \tilde{Y}\left(X_{t-1}, W_t\right)$. It

is just a matter of algebraic operations to determine that

$$
\begin{aligned}
\tilde{Y}\left(X_{t-1}, W_t\right) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \tilde{\Gamma}_0 + \tilde{\Gamma}_1 X_{1,t-1} + \tilde{\Gamma}_2 X_{2,t-1} + \tilde{\Gamma}_3\left(X_{1,t-1} \otimes X_{1,t-1}\right) \\
&\quad + \tilde{\Psi}_0 W_t + \tilde{\Psi}_1\left(X_{1,t-1} \otimes W_t\right) + \tilde{\Psi}_2\left(W_t \otimes W_t\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Gamma}_0 &= \bar{\Gamma}_0 + \bar{\Gamma}_1 \Theta_{10} + \bar{\Gamma}_2 \Theta_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{10}\right) && (48)\\
\tilde{\Gamma}_1 &= \bar{\Gamma}_1 \Theta_{11} + \bar{\Gamma}_2 \Theta_{21} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{11} + \Theta_{11} \otimes \Theta_{10}\right) \\
\tilde{\Gamma}_2 &= \bar{\Gamma}_2 \Theta_{22} \\
\tilde{\Gamma}_3 &= \bar{\Gamma}_2 \Theta_{23} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Theta_{11}\right) \\
\tilde{\Psi}_0 &= \bar{\Gamma}_1 \Lambda_{10} + \bar{\Gamma}_2 \Lambda_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Lambda_{10} + \Lambda_{10} \otimes \Theta_{10}\right) \\
\tilde{\Psi}_1 &= \bar{\Gamma}_2 \Lambda_{21} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Lambda_{10} + \left[\Lambda_{10} \otimes \left(\Theta_{11}\right)_{\bullet j}\right]_{j=1}^{n}\right) \\
\tilde{\Psi}_2 &= \bar{\Gamma}_2 \Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10} \otimes \Lambda_{10}\right).
\end{aligned}
$$

This set of equations defines the mapping

$$
\tilde{\mathcal{P}} = \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}\right).
$$

A.4 Iterative formulas

We can write the conditional expectation in (43) recursively as

$$
\log E\left[M_t \mid \mathcal{F}_0\right] = \log E\left[\exp\left(Y_1 - Y_0\right) E\left[\frac{M_t}{M_1} \mid \mathcal{F}_1\right] \mid \mathcal{F}_0\right].
$$

Given the mappings $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, we can therefore express the coefficients $\bar{\mathcal{P}}$ in (43) using the recursion

$$
\bar{\mathcal{P}}_t = \bar{\mathcal{E}}\left(\mathcal{P} + \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}_{t-1}\right)\right)
$$

where the addition is by coefficients and all coefficients in $\bar{\mathcal{P}}_0$ are zero matrices.

A.5 Coefficients Φ_i^*

In the above calculations, we constructed a recursion for the coefficients in the computation of the conditional expectation of the multiplicative functional M. A single iteration of this recursion can be easily adapted to compute the coefficients Φ_i^*, $i = 0, \ldots 3$, in the conditional expectation in equation (24) for an arbitrary function $\log f\left(x\right)$.

1. Associate $\log f\left(x_{t+1}\right) = \overline{Y}\left(x_{t+1}\right)$ from equation (46), i.e., set $\overline{\Gamma}_i$, $i = 0, \ldots, 3$, in equation (46) equal to the desired Φ_i from equation (23). These are the coefficients in set $\overline{\mathcal{P}}$.

2. Apply the mapping $\widetilde{\mathcal{E}}\left(\overline{\mathcal{P}}\right)$, i.e., compute $\widetilde{\Gamma}_i$, $i = 0, \ldots, 3$ and $\widetilde{\Psi}_i$, $i = 0, 1, 2$ using (48). This yields the function $\log \widetilde{f}\left(x_t, w_{t+1}\right) \equiv \log f\left(x_{t+1}\right)$, with coefficient set $\widetilde{\mathcal{P}}$

3. Add to these coefficients $\widetilde{\Gamma}_i$ and $\widetilde{\Psi}_i$ the corresponding coefficients Γ_i and Ψ_i of $Y_{t+1} - Y_t$ from equation (42), i.e., form coefficient set $\mathcal{P} + \widetilde{\mathcal{E}}\left(\overline{\mathcal{P}}\right)$.

4. Apply the mapping $\overline{\mathcal{E}}\left(\mathcal{P} + \widetilde{\mathcal{E}}\left(\overline{\mathcal{P}}\right)\right)$, i.e., compute (47) where on the right hand side the coefficients Γ_i and Ψ_i (coefficient set $\mathcal{P}$) are replaced with coefficients computed in the previous step, i.e., set $\mathcal{P} + \widetilde{\mathcal{E}}\left(\overline{\mathcal{P}}\right)$.

5. The resulting coefficients $\overline{\Gamma}_i$, $i = 0, \ldots, 3$, are the desired coefficients Φ_i^*.

B   Shock elasticity calculations

In this appendix, we provide details on some of the calculations underlying the derived shock elasticity formulas for the convenient functional form from Section 5.1.1. In particular we show, using a complete-the-squares argument, that under the transformed measure generated by the random variable $L_{1,t}$ from (25) the shock W_1 remains normally distributed with a covariance matrix:

$$\widetilde{\Sigma}_t = \left[I_k - 2\,\mathrm{sym}\left(\mathrm{mat}_{k,k}\left[\Psi_2 + \Phi_{2,t-1}^*\Lambda_{22} + \Phi_{3,t-1}^*\left(\Lambda_{10} \otimes \Lambda_{10}\right)\right]\right)\right]^{-1}.$$

where I_k is the identity matrix of dimension k.[23] We suppose that this matrix is positive definite. The conditional mean vector for W_1 under the change of measure is:

$$\widetilde{E}\left[W_1 \mid X_0 = x\right] = \widetilde{\Sigma}_t\left[\mu_{t,0} + \mu_{t,1}x_1\right],$$

where $\widetilde{E}$ is the expectation under the change of measure and the coefficients $\mu_{t,0}$ and $\mu_{t,1}$ are given in the following derivation.

Thus the shock elasticity is given by:

$$\begin{aligned}
\varepsilon\left(x, t\right) &= \nu(x) \cdot E\left[L_{1,t}W_1 \mid X_0 = x\right] \\
&= \nu(x)'\widetilde{\Sigma}_t\left[\mu_{t,0} + \mu_{t,1}x_1\right].
\end{aligned}$$

The shock elasticity function in this environment depends on the first component, x_1, of the state vector. Recall from (21) that this component has linear dynamics. The coefficient matrices for the evolution of the second component, x_2, nevertheless matter for the shock elasticities even though these elasticities do not depend on this component of the state vector.

B.1   Shock elasticities under the convenient functional form

To calculate the shock elasticities in Section 5.1.1, utilize the formulas derived in Appendix A to deduce the one-period change of measure

$$\log L_{1,t} = \log M_1 + \log E\left(\frac{M_t}{M_1} \mid X_1\right) - \log E\left[M_1 E\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right].$$

In particular, following the set of formulas (48), define

$$\begin{aligned}
\mu_{0,t} &= \left[\Psi_1 + \Phi_{1,t-1}^*\Lambda_{1,0} + \Phi_{2,t-1}^*\Lambda_{20} + \Phi_{3,t-1}^*\left(\Theta_{10} \otimes \Lambda_{10} + \Lambda_{10} \otimes \Theta_{10}\right)\right]' \\
\mu_{1,t} &= \mathrm{mat}_{k,n}\left[\Psi_1 + \Phi_{2,t-1}^*\Lambda_{21} + \Phi_{3,t-1}^*\left(\Theta_{11} \otimes \Lambda_{10} + \left[\Lambda_{10} \otimes (\Theta_{11})_{\bullet j}\right]_{j=1}^n\right)\right] \\
\mu_{2,t} &= \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(\Psi_2 + \overline{\Gamma}_2\Lambda_{22} + \overline{\Gamma}_3\left(\Lambda_{10} \otimes \Lambda_{10}\right)\right)\right].
\end{aligned}$$

---

[23]This formula uses the result that $(\Lambda_{10}W_1) \otimes (\Lambda_{10}W_1) = (\Lambda_{10} \otimes \Lambda_{10})(W_1 \otimes W_1)$.

Then it follows that

$$
\begin{aligned}
\log L_{1,t} \;=\;& \left(\mu_{0,t} + \mu_{1,t} X_{1,0}\right)' W_1 + \left(W_1\right)' \mu_{2,t} W_1 - \\
& -\frac{1}{2} \log E\left[\exp\left(\left(\mu_{0,t} + \mu_{1,t} X_{1,0}\right)' W_1 + \left(W_1\right)' \mu_{2,t} W_1\right) \mid \mathcal{F}_0\right].
\end{aligned}
$$

Expression (45) then implies that

$$
\begin{aligned}
E\left[L_{1,t} W_1 \mid \mathcal{F}_0\right] \;=\;& \widetilde{E}\left[W_1 \mid \mathcal{F}_0\right] = \\
=\;& \left(I_k - 2\mu_{2,t}\right)^{-1}\left(\mu_{0,t} + \mu_{1t} X_{1,0}\right).
\end{aligned}
$$

The variance of W_1 under the $\widetilde{\cdot}$ measure satisfies

$$
\widetilde{\Sigma}_t = \left(I_k - 2\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(\Psi_2 + \bar{\Gamma}_2 \Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10} \otimes \Lambda_{10}\right)\right)\right]\right)^{-1}.
$$

B.2 Approximation of the shock elasticity function

In Section 5.3.1, we constructed the approximation of the shock elasticity function $\varepsilon\left(x,t\right)$. The first-order approximation is constructed by differentiating the elasticity function under the perturbed dynamics

$$
\varepsilon_1\left(X_{1,0},t\right) = \left.\frac{d}{d\mathsf{q}}\, \nu(X_0\left(\mathsf{q}\right)) \cdot \frac{E\left[M_t\left(\mathsf{q}\right) W_1 \mid X_0 = x\right]}{E\left[M_t\left(\mathsf{q}\right) \mid X_0 = x\right]}\right|_{\mathsf{q}=0} = \nu\left(\bar{x}\right) \cdot E\left[Y_{1,t} W_1 \mid X_0 = x\right].
$$

The first-derivative process $Y_{1,t}$ can be expressed in terms of its increments, and we obtain a state-independent function

$$
\varepsilon_1\left(t\right) = \nu\left(\bar{x}\right) \cdot E\left[\sum_{j=1}^{t-1} \kappa_x\left(\psi_x\right)^{j-1} \psi_w + \kappa_w\right]'
$$

where $\kappa_x, \psi_x, \kappa_w, \psi_w$ are derivative matrices evaluated at the steady state $\left(\bar{x}, 0\right)$.

Continuing with the second derivative, we have

$$
\begin{aligned}
\varepsilon_2\left(X_{1,0}, X_{2,0}, t\right) \;=\;& \left.\frac{d^2}{d\mathsf{q}^2}\, \nu(X_0\left(\mathsf{q}\right)) \cdot \frac{E\left[M_t\left(\mathsf{q}\right) W_1 \mid X_0 = x\right]}{E\left[M_t\left(\mathsf{q}\right) \mid X_0 = x\right]}\right|_{\mathsf{q}=0} = \\
=\;& \nu\left(\bar{x}\right) \cdot \left\{E\left[\left(Y_{1,t}\right)^2 W_1 + Y_{2,t} W_1 \mid \mathcal{F}_0\right] - 2 E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right] E\left[Y_{1,t} \mid \mathcal{F}_0\right]\right\} + \\
& +2\left[\frac{\partial \nu}{\partial x'}\left(\bar{x}\right)\right] X_{1,0} \cdot E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right].
\end{aligned}
$$

However, notice that

$$
\begin{aligned}
E\left[\left(Y_{1,t}\right)^2 W_1 \mid \mathcal{F}_0\right] \;=\;& 2\left(\sum_{j=0}^{t-1} \kappa_x\left(\psi_x\right)^j X_{1,0}\right)\left(\sum_{j=1}^{t-1} \kappa_x\left(\psi_x\right)^{j-1} \psi_w + \kappa_w\right)' \\
E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right] \;=\;& \left(\sum_{j=1}^{t-1} \kappa_x\left(\psi_x\right)^{j-1} \psi_w + \kappa_w\right)' \\
E\left[Y_{1,t} \mid \mathcal{F}_0\right] \;=\;& \sum_{j=0}^{t-1} \kappa_x\left(\psi_x\right)^j X_{1,0}
\end{aligned}
$$

50

and thus
$$E\left[(Y_{1,t})^2 W_1 \mid \mathcal{F}_0\right] - 2E\left[Y_{1,t}W_1 \mid \mathcal{F}_0\right] E\left[Y_{1,t} \mid \mathcal{F}_0\right] = 0.$$

The second-order term in the approximation of the shock elasticity function thus simplifies to

$$\varepsilon_2\left(X_{1,0}, X_{2,0}, t\right) = \nu\left(\bar{x}\right) \cdot E\left[Y_{2,t}W_1 \mid \mathcal{F}_0\right] + 2\left[\frac{\partial \nu}{\partial x'}\left(\bar{x}\right)\right] X_{1,0} \cdot E\left[Y_{1,t}W_1 \mid \mathcal{F}_0\right].$$

The expression for the first term on the right-hand side is

$$
\begin{aligned}
E\left[Y_{2,t}W_1 \mid \mathcal{F}_0\right] &= E\left[\sum_{j=0}^{t-1}\left(Y_{2,j+1} - Y_{2,j}\right)W_1 \mid \mathcal{F}_0\right] = 2\mathrm{mat}_{k,n}\left(\kappa_{xw}\right)X_{1,0} + \\
&\quad + 2\sum_{j=1}^{t-1}\left[\psi'_w\left(\psi'_x\right)^{j-1}\mathrm{mat}_{n,n}\left(\kappa_{xx}\right)\left(\psi_x\right)^j + \mathrm{mat}_{k,n}\left[\kappa_x\left(\psi_x\right)^{j-1}\psi_{xw}\right]\right]X_{1,0} \\
&\quad + 2\sum_{j=1}^{t-1}\sum_{k=1}^{j-1}\left[\psi'_w\left(\psi'_x\right)^{k-1}\mathrm{mat}_{n,n}\left[\kappa_x\left(\psi_x\right)^{j-k-1}\psi_{xx}\right]\left(\psi_x\right)^k\right]X_{1,0}.
\end{aligned}
$$

To obtain this result, notice that repeated substitution for $Y_{1,j+1} - Y_{1,j}$ into the above formula yields a variety of terms but only those containing $X_{1,0} \otimes W_1$ have a nonzero conditional expectation when interacted with W_1.

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