# "Assessing DSGE Model Nonlinearities "

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April 2014

# Motivation

- Identify nonlinearities and evaluate nonlinear DSGE
  - State-space model

$$S_t = \Gamma\left(S_{t-1}, w_{t-1}; \theta\right)$$

$$Y_t^e = M\left(S_t, v_t; \theta\right)$$

  - Statistical model

$$Y_t^s = f\left(Y_{t-1}^s, u_t\right)$$

- First order approximation
  - State-space model

$$S_t^1 = \Gamma^1\left(\theta\right) S_{t-1} + H(\theta) w_t$$

$$Y_t^e = A\left(\theta\right) + B\left(\theta\right) S_t + v_t$$

  - Statistical model

$$Y_t^s = C Y_{t-1}^s + u_t$$

# Motivation

- What reference model for second order approxia-tion?
  - QAR
- How to use this model to evaluate DSGE?
  - Posterior predictive checks

# Quadratic Autoregressive Model (QAR)

- Let

$$y_t^* = f\left(y_{t-1}^*, \omega u_t\right) \qquad \text{where } u_t \sim N(0,1)$$

- Second order approximation

$$y_t = y_t^0 + \omega y_t^{(1)} + \omega^2 y_t^{(2)}$$

- So that

$$\begin{aligned} y_t^* - \bar{y} &= f_y\left(y_{t-1}^* - \bar{y}\right) + f_u \omega u_t \\ &+ \frac{1}{2} f_{y,y}\left(y_{t-1}^* - \bar{y}\right)^2 + f_{y,u}\left(y_{t-1}^* - \bar{y}\right)\omega u_t \\ &+ \frac{1}{2} f_{u,u}\left(\omega u_t\right)^2 + \text{higher order terms} \end{aligned}$$

- Substitute $y_t$ and match coefficients

# QAR

- The resulting approximation is

$$y_t = \phi_0 + \phi_1 (y_{t-1} - \bar{y}) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1}) \sigma u_t + \frac{1}{2}\phi_3 \omega^2 u_t^2$$

$$s_t = \phi_1 s_{t-1} + \sigma u_t$$

- Unique steady state and non-explosive if $|\phi_1| < 1$

- Not true for "standard" approximation

$$\hat{y}_t - \bar{y} = \phi_1 (\hat{y}_t - \bar{y}) + \phi_2 (\hat{y}_t - \bar{y})^2$$

# Why QAR?

- State dependent IRFs

$$IRF_t(h) = E_t[y_{t+h}|u_t = 1] - E_t[y_{t+h}]$$

$$IRF_t(0) = \sigma(1 + \gamma s_{t-1})$$

$$IRF_t(1) = \sigma\left(\phi_1(1 + \gamma s_{t-1}) + 2\phi_1\phi_2\sqrt{1 - \phi_1^2}s_{t-1}\right)$$

- Conditional Heteroskedasticity

$$V_{t-1}[y_t] = (1 + \gamma s_{t-1})^2\sigma^2$$

# How to use QAR?

- Estimate QAR
- Estimate 2nd order approximation to DSGE
- Use posterior on DSGE parameters to get a posterior predictive distribution on QAR estimates
- Check how far the actual QAR estimate lies in the tail of this distribution

# QAR: Estimation

- Computing

$$p\left(Y_{0:T}, \theta, s_0\right) = p\left(Y_{1:T} | y_{0,} s_0, \theta\right) p\left(y_0, s_0 | \theta\right) p\left(\theta\right)$$

- Factorize likelihood

$$p\left(Y_{1:T} | y_{0,} s_0, \theta\right) = \prod_{t=1}^{T} p\left(y_t | y_{0:t-1,} s_0, \theta\right)$$

- Computed recursively using

$$p\left(y_t | y_{t-1}, s_{t-1}\right) \sim N \qquad s_t = g\left(y_t, y_{t-1}, s_{t-1}\right)$$

# QAR Estimation

- Initialization

$$p\left(y_0, s_0|\theta\right) = N\left(\left[\begin{array}{c}\mu_y \\ \mu_d\end{array}\right], \left[\begin{array}{cc}\Sigma_{yy} & \Sigma_{ys} \\ \Sigma_{sy} & \Sigma_{ss}\end{array}\right]\right)$$

- Substitute in steady state at $t = -T^*$

- Find

$$E\left(s_j\right), E\left(y_j\right), V\left(s_j\right), V\left(y_j\right), cov\left(s_j, y_j\right), cov(s_j^2, y_j), V\left(s_j^2\right)$$

as a function of their lagged values using the QAR law of motions

# QAR Estimation

- Priors:

| | GDP Growht | Wage Growth | Inflation | Fed Funds Rate |
|---|---|---|---|---|
| $\phi_0$ | $N\left(.48, 2\right)$ | $N\left(1.18, 2\right)$ | $N\left(2.38, 2\right)$ | $N\left(2.50, 2\right)$ |
| $\phi_1$ | $N^T\left(.36, .5\right)$ | $N^T\left(-.02, .5\right)$ | $N^T\left(0.00, .5\right)$ | $N^T\left(0.66, .5\right)$ |
| $\sigma$ | $IG\left(1.42, 4\right)$ | $IG\left(.82, 4\right)$ | $IG\left(1.87, 4\right)$ | $IG\left(.58, 4\right)$ |
| $\phi_2$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ |
| $\gamma$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ | $N\left(0, 0.1\right)$ |

- Pre-sample information to parametrize priors

# QAR Estimation

- RWM Algorithm:

- Use prior to get a Cov matrix for parameters $\Omega$

- Produce 100k draws using proposal density

$$\hat{\theta} = \theta_t + U_t \qquad U_t \sim N(0, \Omega)$$

- Use last 50k to compute $\Omega'$

- Produce 60k draws using new proposal density

$$\hat{\theta} = \theta_t + U_t' \qquad U_t' \sim N\left(0, \Omega'\right)$$

# DSGE

- New Keynesian DSGE with asymmetric price and wage adjustment costs
- 4 exogenous shocks: tfp; markup; government; monetary policy.

- Approximate solution using "standard" method

- Bayesian estimation using RWM and particle filter

# Particle Filter

- The goal is to approximate

$$p\left(y_t \left| Y^{t-1}, \theta\right.\right) = \int p\left(y_t \left| s_t, \theta\right.\right) p\left(s_t \left| Y^{t-1}, \theta\right.\right) ds_t$$

  - Start from $p\left(s_0 \left| \theta\right.\right)$ to draw $\left\{s_0^i\right\}_{i=1}^N$; Assume we have $\left\{s_{t-1}^i\right\}_{i=1}^N$ which approximate $p\left(s_{t-1} \left| Y_{t-1}, \theta\right.\right)$
  - $p\left(s_t \left| Y_{t-1}, \theta\right.\right)$ is approximated by

$$\begin{aligned} p\left(s_t \left| Y_{t-1}, \theta\right.\right) &= \int p\left(s_t \left| s_{t-1}, \theta\right.\right) p\left(s_{t-1} \left| Y^{t-1}, \theta\right.\right) ds_t \\ &\approx \frac{1}{N} \sum p\left(s_t \left| s_{t-1}^i, \theta\right.\right) \end{aligned}$$

# Particle Filter

- – Drawing $\left\{\tilde{s}_t^i\right\}_{i=1}^N$ from $p\left(s_t \left| s_{t-1}^i, \theta\right.\right)$ approxiamates $p\left(s_t \left| Y_{t-1}, \theta\right.\right)$ hence

  $$p\left(y_t \left| Y^{t-1}, \theta\right.\right) = \int p\left(y_t \left| s_t, \theta\right.\right) p\left(s_t \left| Y^{t-1}, \theta\right.\right) ds_t \approx \frac{1}{N} \sum p\left(y_t \left| \tilde{s}_t^i, \theta\right.\right)$$

  – Finally get an approximation $\left\{s_t^i\right\}_{i=1}^N$ of $p\left(s_t \left| Y_t, \theta\right.\right)$ by drawing with replacement from $\left\{\tilde{s}_t^i\right\}_{i=1}^N$ with pmf given by

  $$\pi_t^i = \frac{p\left(y_t \left| \tilde{s}_t^i, \theta\right.\right)}{\sum p\left(y_t \left| \tilde{s}_t^i, \theta\right.\right)}$$

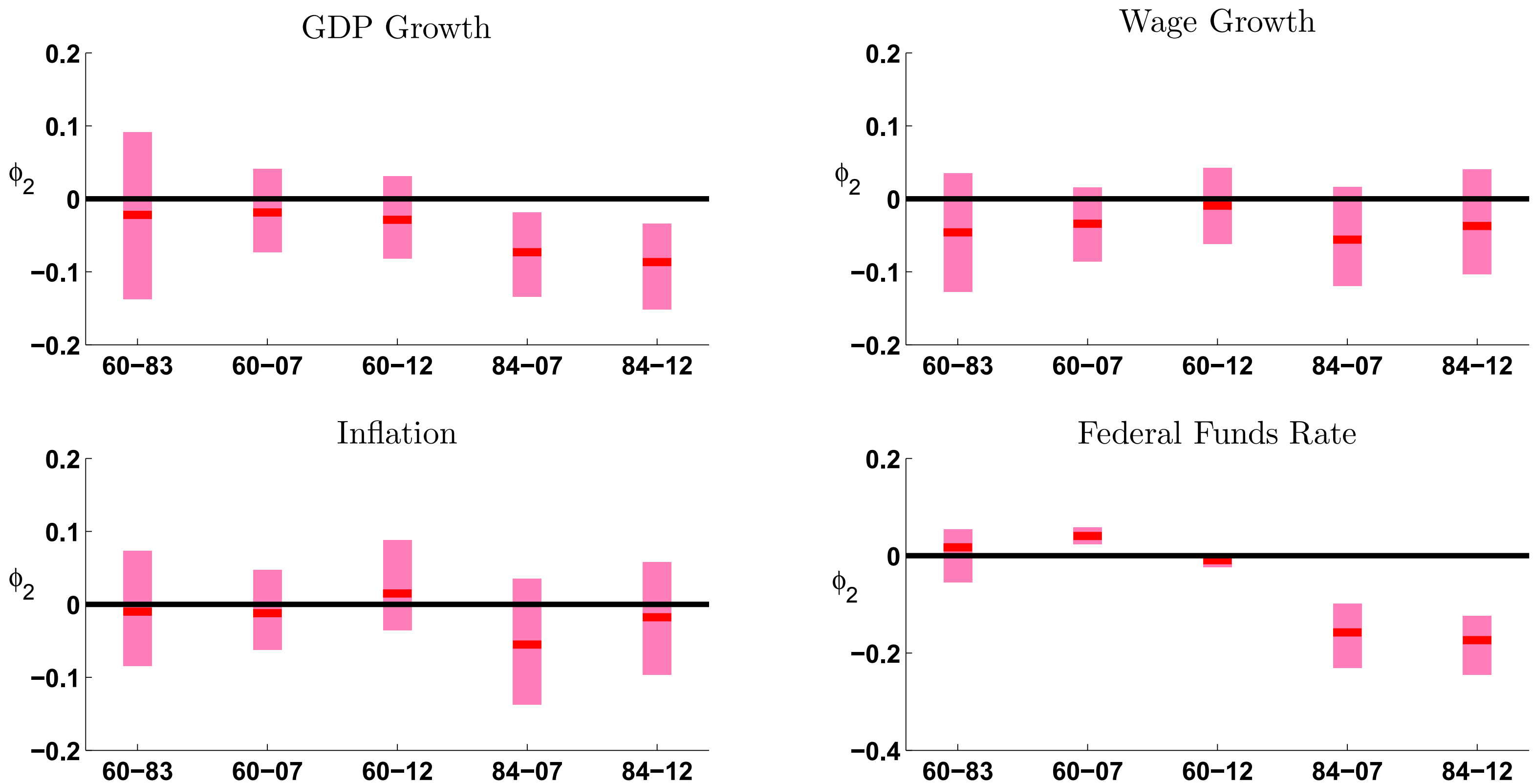
# Posterior predictive checks

- Draw $\theta^i$ from posterior of the DSGE parameters

- Simulate Data from the DSGE $\left\{ Y^i_{-T^*:T} \right\}$ and obtain median estimate of QAR parameters $S^i$

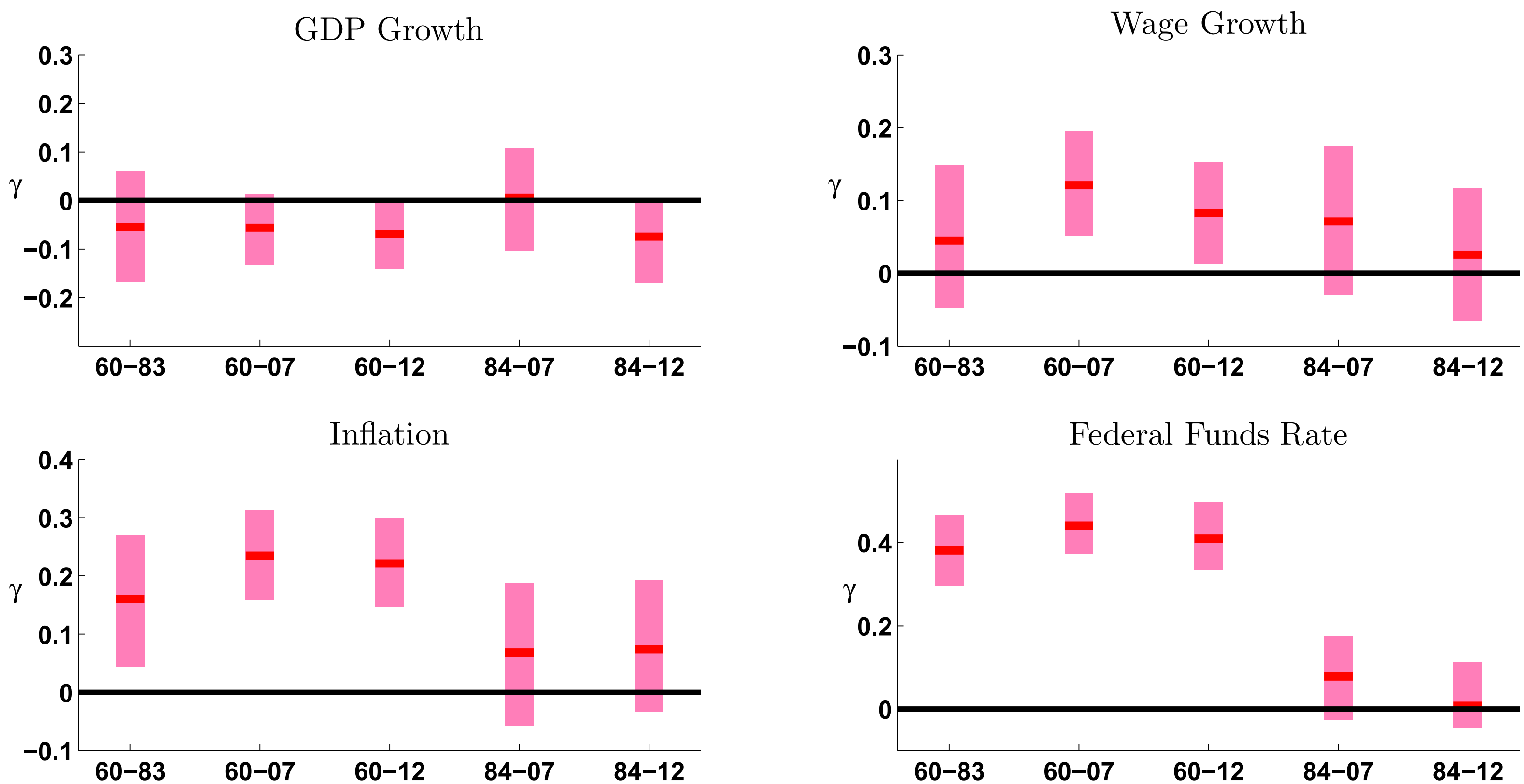
- Examine how far the median estimate from actual US data lie in the tail of the empirical distribution of $S^i$

# Estimation of QAR(1,1) Model on U.S. Data – $\Phi_2$



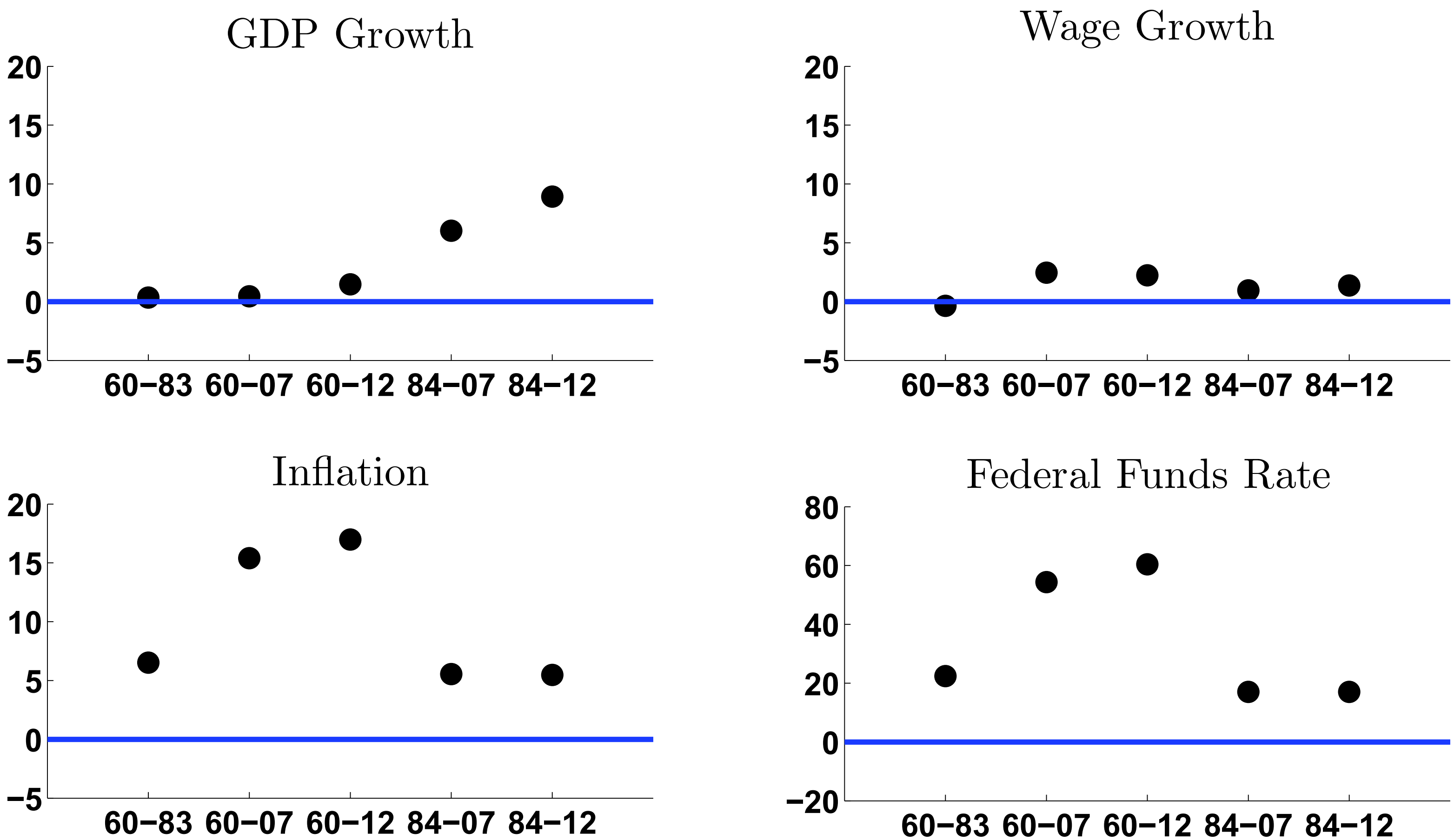

$$
\begin{aligned}
y_t &= \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t \\
s_t &= \phi_1 s_{t-1} + \sigma u_t \qquad\qquad u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)
\end{aligned}
$$

# Estimation of QAR(1,1) Model on U.S. Data – $\gamma$

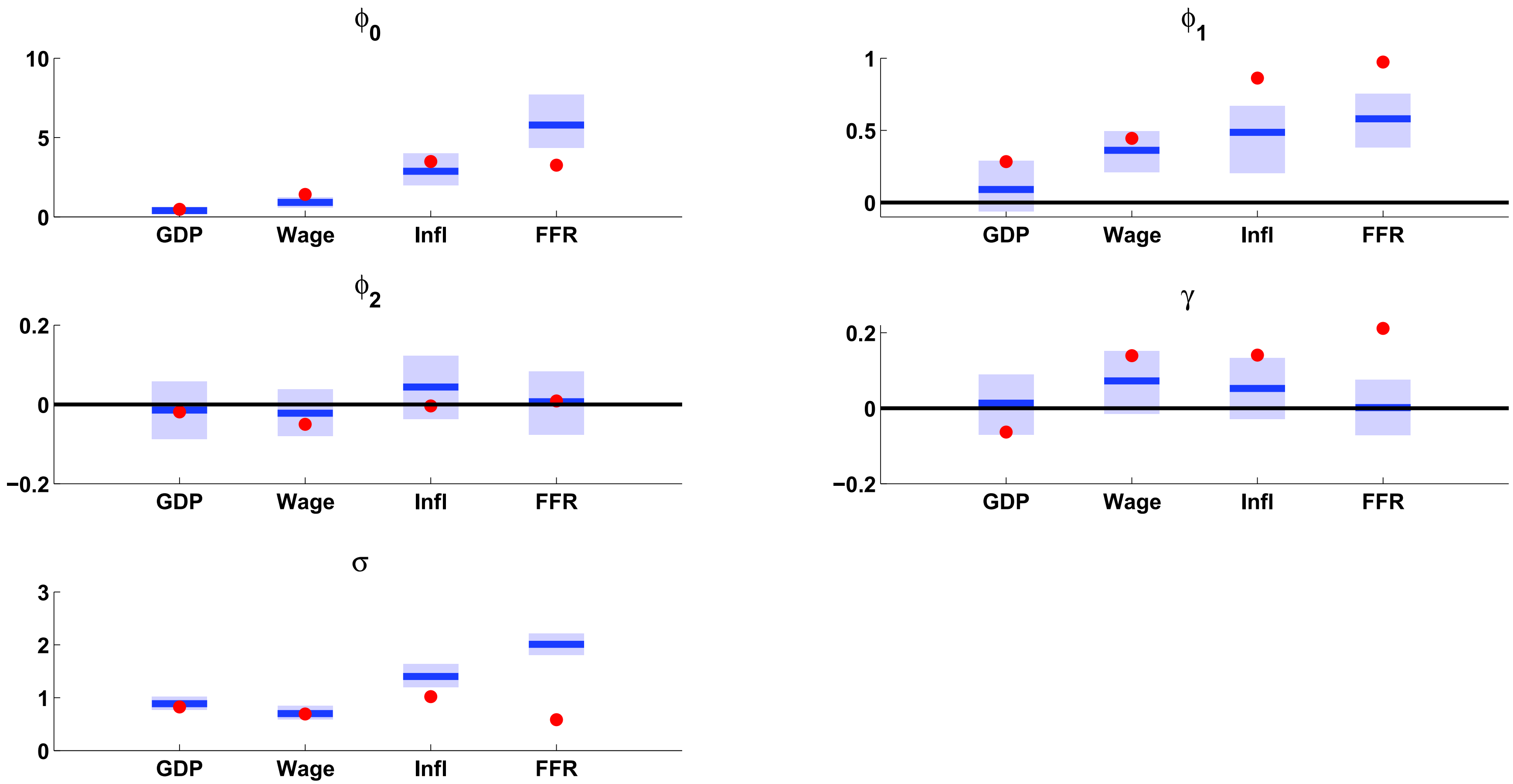



$$
\begin{aligned}
y_t &= \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t \\
s_t &= \phi_1 s_{t-1} + \sigma u_t \qquad\qquad u_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)
\end{aligned}
$$

# Log Marginal Data Density Differentials: QAR(1,1) versus AR(1)

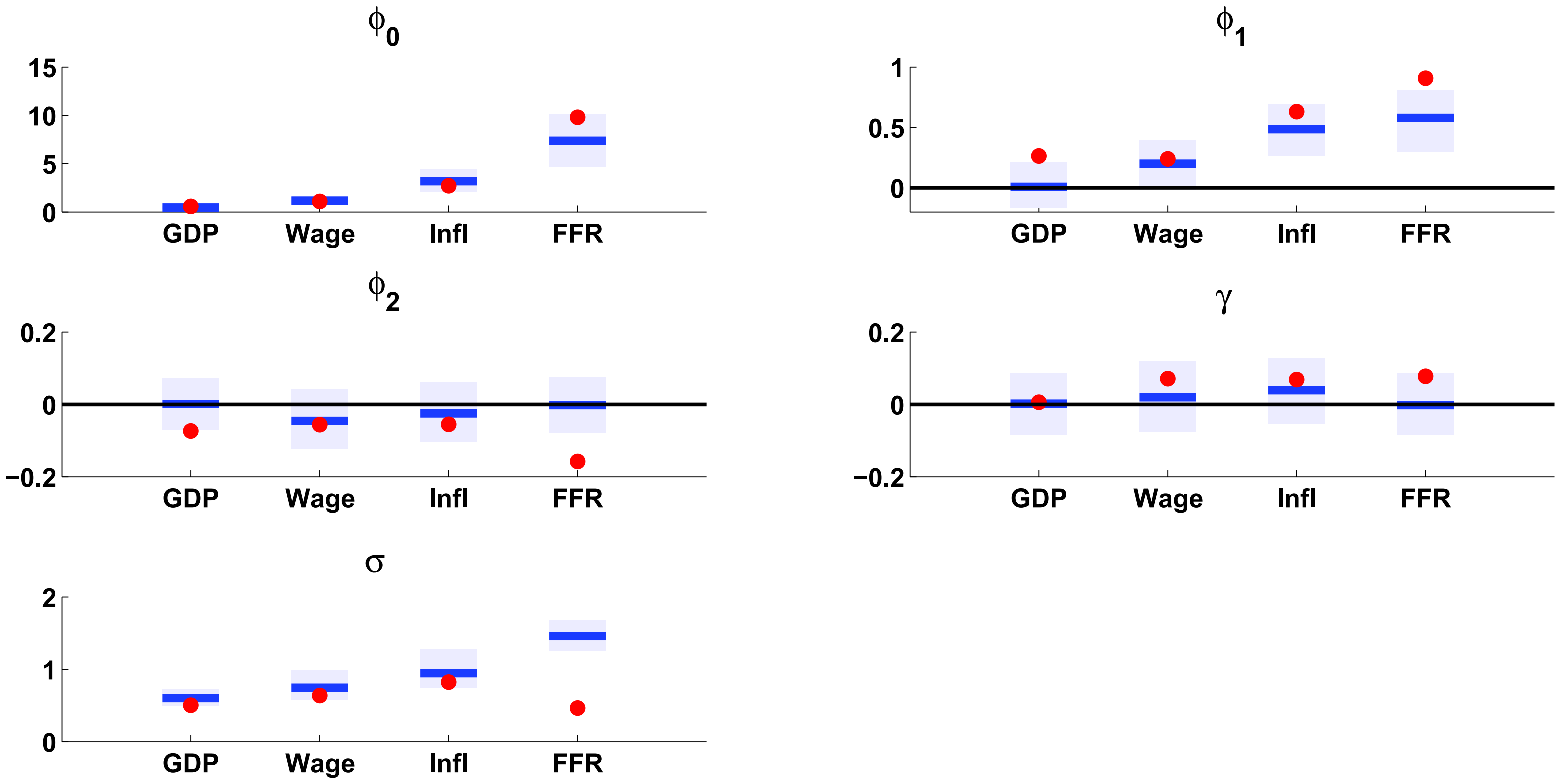


# Posterior Predictive Checks: 1960-2007 Sample



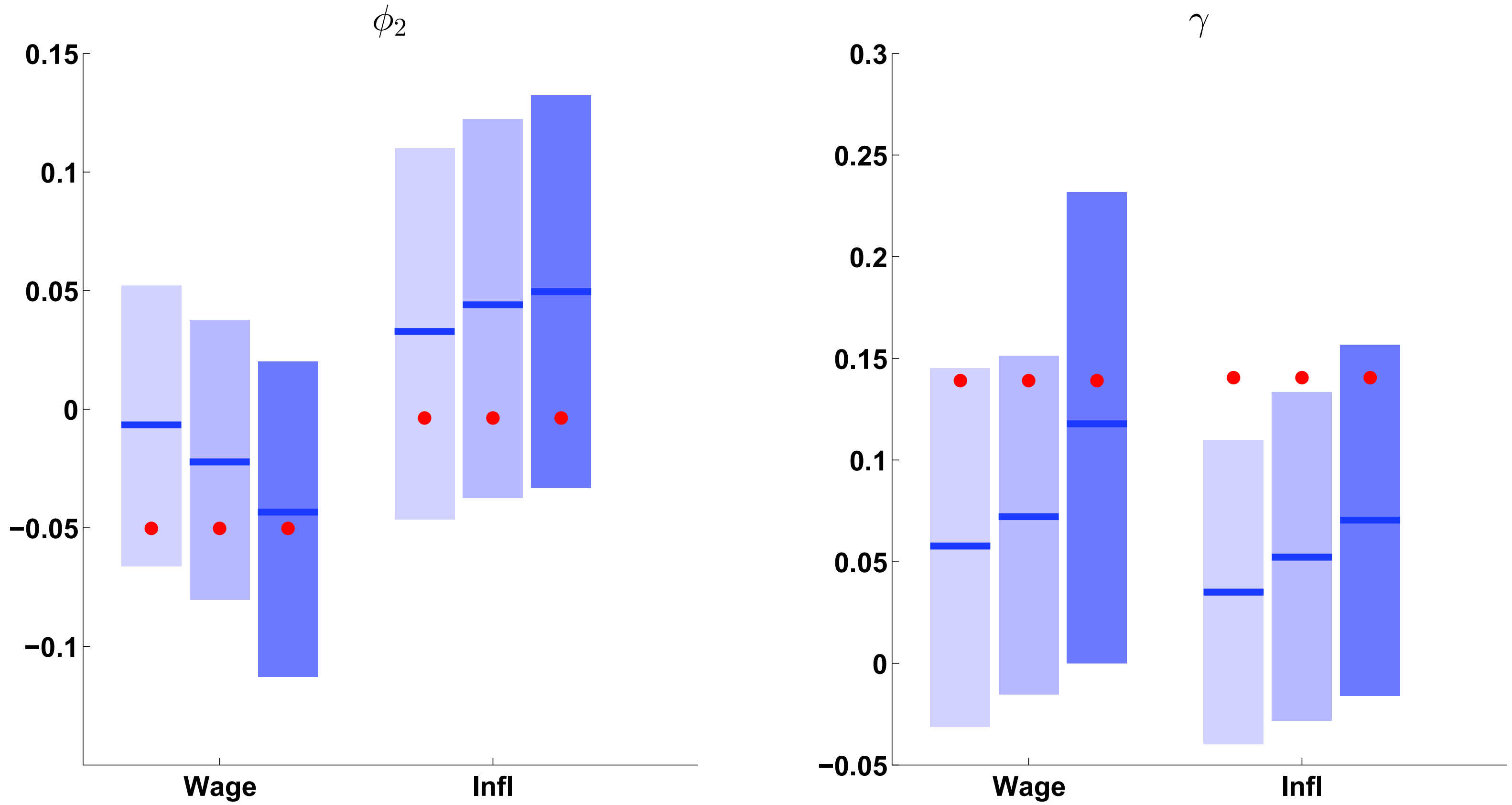

- QAR estimates from actual and model-generated data are similar.
- Only interest rates exhibit noticeable differences.
- Except for wage and inflation $\hat{\gamma}$, nonlinearities are generally weak.

# Posterior Predictive Checks: 1984-2007 Sample




- Model does not generate nonlinearity ($\hat{\phi}_2$) in GDP dynamics.

# Effect of Adjustment Costs on Nonlinearities: 1960-2007 Sample




No asymmetric costs is $\psi_p = \psi_w = 0$ (light blue); high asymmetric costs is $\psi_p = \psi_w = 300$ (dark blue). Large dots correspond to posterior median estimates based on U.S. data.