Doubts and Variability

Authors: Rhys Bidder and Matthew E. Smith

Presentation: Dan Greenwald

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Motivation

- Paper considers asset-pricing implications of model uncertainty.
- Estimates underlying endowment process, and considers multiplier preferences given these "true" models.
- Investigates effect of model uncertainty on Hansen-Jagannathan bounds in the presence of stochastic volatility.
- Characterizes worst-case probability distribution and detection error probabilities from the robust agent's perspective.

Agenda

- First, estimate parameters of the consumption growth process using MCMC sampler.
- Given these estimates, and a solution to the agent's optimization problem, we can do all the asset pricing, etc.
- However, may also be interested in features of the robust control problem:
  1. What are the properties of the worst case probability distribution?
  2. What is the link between the consumption growth process and detection error probability?
- Calculating these objects will require further MCMC sampling, given parameters of endowment process.

Consumption Process

- Homoskedastic version:

$$\Delta \log(C_{t+1}) = \phi + \sigma \varepsilon_{t+1}$$
$$\varepsilon_{t+1} \sim N(0,1)$$

- Stochastic volatility version:

$$\Delta \log(C_{t+1}) = \phi + \sigma \exp(v_{t+1}) \varepsilon_{1,t+1}$$
$$v_{t+1} = \lambda v_t + \tau \varepsilon_{2,t+1}$$
$$\begin{pmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{pmatrix} \sim N(0, I)$$

- Consumption is observable, so we can estimate the endowment process without making any assumptions on preferences.

Estimating the Consumption Process

- Estimation using Bayesian methods.
- Priors:

Parameter	Description	Prior
ϕ	Mean Consumption Growth	Uniform [0, 1]
σ	Non-Stoch. Consumption Growth Vol.	Uniform [0, 1]
τ	SV Innovation Volatility	Uniform [0, 1]
λ	SV Persistence	Uniform [-1, 1]

- Estimation method:
  - Homoskedastic: Random Walk Metropolis-Hastings Algorithm.
    - Alternatives: could have used conjugate prior and sampled directly, or done importance sampling here.
  - SV: Particle Marginal Metropolis-Hastings Algorithm.

Review of Bayesian Econometrics

- For notation, let $\xi = (\phi, \sigma)'$ be the vector of parameters, and let y denote the data $(\Delta \log(C_1), \ldots, \Delta \log(C_T))$.
- Want to draw from the posterior distribution $p(\xi|y)$.
- By Bayes' rule, we have $p(\xi|y) \propto p(y|\xi)p(\xi)$.
- Prior $p(\xi)$ is known by construction.
- Likelihood $p(y|\xi)$ is known given data:

$$p(y|\xi) = (2\pi)^{-T/2}\sigma^{-T} \exp\left\{ -\frac{1}{2}\sigma^{-2} \sum_{t=1}^{T} (\log(C_t) - \phi)^2 \right\}.$$

Metropolis-Hastings Algorithm

1. Given current draw ξ^j, choose candidate ξ^* from a proposal density $q(\xi^*; \xi^j)$.
   - Random walk proposal: $\xi^* = \xi^j + \eta$, for $\mathbb{E}[\eta] = 0$.
2. Calculate acceptance probability

$$\alpha = \min \left\{ \frac{p(\xi^*|y)/q(\xi^*; \xi_j)}{p(\xi_j|y)/q(\xi_j; \xi^*)}, 1 \right\} = \min \left\{ \frac{p(y|\xi^*)p(\xi^*)/q(\xi^*; \xi_j)}{p(y|\xi_j)p(\xi_j)/q(\xi_j; \xi^*)}, 1 \right\}.$$

   - If proposal distribution is symmetric, then

$$\alpha = \min \left\{ \frac{p(y|\xi^*)p(\xi^*)}{p(y|\xi_j)p(\xi_j)}, 1 \right\}$$

3. Set $\xi^{j+1} = \xi^*$ with probability α, set $\xi^{j+1} = \xi^j$ with probability $1 - \alpha$.

Particle Marginal Metropolis-Hastings Algorithm

- In the previous case, we assumed that the likelihood $p(y|\xi)$ is known.
- However, in the SV specification, this is no longer the case.
- Instead, we can calculate an approximation $\hat{p}(y|\xi)$ using a particle filter.
- We can then proceed as before using $\hat{p}(y|\xi^j)$ and $\hat{p}(y|\xi^*)$ in place of $p(y|\xi^j)$ and $p(y|\xi^*)$.

SIR Particle Filter

- A good basic particle filtering algorithm is Sampling Importance Resampling (SIR).
- For notation, let y_t be observable data, and let x_t be latent states. Assume that $p(y_t|x^T) = g(y_t|x_t)$, that $p(x_t|x_{t-1}, \ldots, x_1) = f(x_t|x_{t-1})$, and that $p(x_1) = \mu(x_1)$.
- At $t = 1$:
  - Initialize $x_1^i \sim q_1(x_1|y_1)$ for $i = 1, \ldots, N$, from some proposal density q_1.
  - Compute weights $w_1^i = \dfrac{\mu(x_1^i)g(y_1|x_1^i)}{q_1(x_1^i|y_1)}$, normalized weights $W_1^i \propto w_1^i$.
  - Resample $\{W_1^i, x_1^i\}$ to obtain N equally weighted particles $\bar{x}_1^i$.

SIR Particle Filter

- At $t \geq 2$:
  - Sample $x_t^i \sim q_t(x_t | y_t, \bar{x}_{t-1}^i)$.
  - Compute incremental weights $\alpha_t^i = \dfrac{g(y_t | x_t^i) f(x_t^i | \bar{x}_{t-1}^i)}{q(x_t^i | y_t, \bar{x}_{t-1}^i)}$ and normalized weights $W_t^i \propto \alpha_t^i$.
  - Resample $\{W_t^i, x_t^i\}$ to obtain N equally weighted particles $\bar{x}_t^i$.
- Given output of algorithm, can approximate

$$\hat{p}(y_t | y^{t-1}, \xi) = \sum_{i=1}^{N} W_{t-1}^i \alpha_t^i$$

$$\hat{p}(y | \xi) = \hat{p}(y_t | y^{t-1}, \xi) \cdots \hat{p}(y_2 | y_1, \xi) \hat{p}(y_1 | \xi).$$

- For the SV problem, $x_t = v_t$, $y_t = \Delta \log(C_t)$, use true transition probabilities for v_t as the proposal q.
- See Doucet and Johansen (2008) for further improvements to particle filter, Andrieu, Doucet and Holenstein (2010) for more information about PMCMC.

Multiplier Preferences

- Notation: current state is x, next period's state is $x'(\varepsilon'; x)$.
- Bellman equation:

$$W(x) = \log(C(x)) + \min_{m(\varepsilon;x)\geq 0} \left(\beta \int [m(\varepsilon; x) W(x'(\varepsilon'; x)) + \theta m(\varepsilon; x) \log(m(\varepsilon; x))] p(\varepsilon)\, d\varepsilon \right)$$

- Bellman equation at minimizing m:

$$W(x) = \log(C(x)) - \beta\theta \log \left(\int \exp\left(\frac{-W(x'(\varepsilon'; x))}{\theta} \right) p(\varepsilon)\, d\varepsilon \right)$$

Asset Pricing

- Stochastic discount factor:

$$\Lambda_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\frac{\exp\left(\frac{-W_{t+1}}{\theta}\right)}{\mathbb{E}_t \left[\exp\left(\frac{-W_{t+1}}{\theta}\right) \right]} \right).$$

- Decomposition:

$$\Lambda_{t,t+1} = \Lambda^R_{t,t+1} \Lambda^U_{t,t+1}$$

$$\Lambda^R_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1}$$

$$\Lambda^U_{t,t+1} = \frac{\exp\left(\frac{-W_{t+1}}{\theta}\right)}{\mathbb{E}_t \left[\exp\left(\frac{-W_{t+1}}{\theta}\right) \right]}$$

Asset Pricing

- Authors use third-order perturbations to solve for the value function and the stochastic discount factor $\Lambda_{t,t+1}$.
- Therefore, given the earlier estimates of the endowment process, we can price any asset, check HJ bounds, etc.
- Rest of the paper will characterize the robust agent's problem (worst case distribution, detection error probabilities).

Distorted Expectations

- Reformulation of asset pricing equation:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\Lambda_{t,t+1} R_{t+1}\right] \\ &= \int R(\varepsilon)\cdot\beta\left(\frac{C(x'(\varepsilon';x))}{C(x)}\right)^{-1}\left(\frac{\exp\left(\frac{-W(x'(\varepsilon';x))}{\theta}\right)}{\mathbb{E}_t\left[\exp\left(\frac{-W(x'(\varepsilon';x))}{\theta}\right)\right]}\right) p(\varepsilon)\,d\varepsilon \\ &= \int R(\varepsilon)\cdot\beta\left(\frac{C(x'(\varepsilon';x))}{C(x)}\right)^{-1}\tilde{p}(\varepsilon;x)\,d\varepsilon \\ &= \tilde{E}_t\left[\Lambda^R_{t,t+1} R_{t+1}\right] \end{aligned}$$

- Distorted probability measure:

$$\tilde{p}(\varepsilon;x) = \left(\frac{\exp\left(\frac{-W(x'(\varepsilon';x))}{\theta}\right)}{\mathbb{E}_t\left[\exp\left(\frac{-W(x'(\varepsilon';x))}{\theta}\right)\right]}\right) p(\varepsilon)$$

Distorted Expectations

- Therefore, the agent prices assets as if he or she had log expected utility preferences, but under the probability distribution $\tilde{p}$.
- Distribution $\tilde{p}$ is known as the worst-case distribution.
- This is itself an object of interest: what is the consumption process that the agent has in mind when pricing assets?
- This density does not have a standard form, so we will once again use Monte Carlo methods to sample from it.
- For notation, let s be the deterministic variables in the state x, so that $s' = f(\varepsilon, s)$. (Here $s_t = v_t$).

Sampling the Worst Case Distribution

- Method 1: Random Walk Metropolis-Hastings
- Given $\{\varepsilon_{t-1}^i, s_{t-1}^i\})_{i=1}^N$:
    1. Set $s_t^i = f(\varepsilon_{t-1}^i, s_{t-1}^i)$.
    2. For $i = 1, \ldots, N$:
    3. Draw $\varepsilon_t^* \sim q(\varepsilon^*, \varepsilon_{t-1}^i)$ for some proposal density q.
    4. Set $\varepsilon_t^i = \varepsilon_t^*$ with probability $\min\left\{1, \frac{\tilde{p}(\varepsilon_t^*)/q(\varepsilon_t^*, \varepsilon_{t-1}^i)}{\tilde{p}(\varepsilon_t^{i-1})/q(\varepsilon_t^{i-1}, \varepsilon_t^*)}\right\}$, and set $\varepsilon_t^i = \varepsilon_t^{i-1}$ otherwise (note: incorrect in paper!).
    5. Increment t.
- Can use p distribution as proposal: $q \sim N(0, I)$.
- Alternative to Metropolis-Hastings: could instead use p as a proposal to do importance sampling.

Sampling the Worst Case Distribution

- Method 2: SIR
- Given $\{\bar{\varepsilon}^i_{t-1}, s^i_{t-1}\})^N_{i=1}$:
  1. Set $s^i_t = f(\varepsilon^i_{t-1}, s^i_{t-1})$.
  2. For $i = 1, \ldots, N$:
  3. Draw $\varepsilon^i_t \sim p(\varepsilon_t)$.
  4. Assign weight $w^i_t = \exp\left(\frac{-W(x_t)}{\theta}\right)$.
  5. Resample from $\{\varepsilon^i_t\}^N_{i=1}$ with probability $\propto w^i_t$ to obtain $\{\bar{\varepsilon}^i_t\}^N_{i=1}$.
  6. Increment t.
- Even simpler here because no signal extraction problem.
- Note: could draw $\varepsilon^i_t \sim q(\varepsilon_t)$ for any proposal q, and use weights $w^i_t = \tilde{p}(\varepsilon^i_t)/q(\varepsilon^i_t)$.

Detection Error Probability

- The robustness parameter θ can be associated with a detection error probability relative to the worst-case model, which can be used to discipline the calibration of θ.
- Given two models, detection error probability is the probability that, given data simulated from one model, the other model's likelihood function is larger (with equal weight on which model generated the data).
- To compute detection error given a true model $\mathcal{M}_0$ and an alternative model $\mathcal{M}_1$:
    1. Compute the fraction of simulations generated under the true model for which $p(y|\mathcal{M}_1) > p(y|\mathcal{M}_0)$. Call this r_0.
    2. Compute the fraction of simulations generated under the alternative model for which $p(y|\mathcal{M}_0) > p(y|\mathcal{M}_1)$. Call this r_1.
    3. Then the detection error probabilty is given by $\frac{1}{2}(r_0 + r_1)$.
- Key ingredient in this procedure: $p(y|\mathcal{M})$.

Detection Error Probability

- For a given parameter ξ, we are interested in the detection error probability between p and $\tilde{p}$ (how confident is the agent that the worst-case model is wrong?).
- Need to calculate the true likelihood $p(y|\xi)$ and the worst-case likelihood $\tilde{p}(y|\xi)$.
- For the true likelihood, we can use the SIR particle filter as before.
- For the worst-case likelihood, need to assume that $\tilde{p}(y_t|x_t) = p(y_t|x_t)$.
- In this case, we can again use the SIR particle filter.
- The authors choose the proposal density $q(x_t|x_{t-1}) = p(x_t|x_{t-1})$, where p is the probability under the true model.
- Authors also add measurement error, so that $\Delta \log(C_{t+1}) = \phi + \exp(v_{t+1})\varepsilon_{1,t+1} + \varepsilon_{3,t+1}$, which they say is needed to compute detection error probabilities (why?).

References

- Andrieu, Christophe and Arnaud Doucet and Roman Holenstein, "Particle Markov Chain Monte Carlo Methods", *Journal of the Royal Statistical Society*, Vol. 72, Part 3, pp. 269–342, 2010.
- Bidder, Rhys and Matthew E. Smith, "Doubts and Variability", *Mimeo*, 2011.
- Doucet, Arnaud and Adam M. Johansen, "A Tutorial on Particle Filtering and Smoothing: Fifteen Years Later", *Mimeo*, 2008.
- Hansen, Lars and Thomas J. Sargent, "Fragile Beliefs and the Price of Uncertainty", *Mimeo*, 2010.