# Survival and long-run dynamics with heterogeneous beliefs under recursive preferences*

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**Abstract**

I study the long-run behavior of an economy with two types of agents who differ in their beliefs and are endowed with homothetic recursive preferences of the Duffie–Epstein–Zin type. Contrary to models with separable preferences in which the wealth of agents with incorrect beliefs vanishes in the long run, recursive preference specifications lead to long-run outcomes where both agents survive, or more incorrect agents dominate. I derive analytical conditions for the existence of nondegenerate long-run equilibria in which agents who differ in accuracy of their beliefs coexist in the long run, and show that these equilibria exist for broad ranges of plausible parameterizations when risk aversion is larger than the inverse of the intertemporal elasticity of substitution. The results highlight a crucial interaction between risk sharing, speculative behavior and consumption-saving choice of agents with heterogeneous beliefs, and the role of equilibrium prices in shaping long-run outcomes.

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# 1 Introduction

A growing body of empirical evidence documents systematic and persistent differences in portfolio returns and saving rates across agents. The evidence calls for theoretical models to analyze the sources of these differences and consequences for general equilibrium prices and evolution of the wealth distribution in the economy. This paper analyzes the implications of heterogeneity in agents' beliefs as one plausible factor contributing to these phenomena. Taking belief heterogeneity as given, I study the determinants of long-run wealth dynamics in a class of equilibrium economies populated by agents endowed with nonseparable recursive preferences.

These preferences, axiomatized by Kreps and Porteus (1978), and developed by Epstein and Zin (1989) and Weil (1990) in discrete time and by Duffie and Epstein (1992b) in continuous time, allow one to disentangle the risk aversion with respect to intratemporal gambles from the intertemporal elasticity of substitution (IES), and include the separable, constant relative risk aversion (CRRA) utility as a special case. Thanks to the additional degree of flexibility and the resulting ability to provide a better account of the patterns observed in asset return data, this class of preferences became the workhorse model used in the asset pricing literature.

I provide a complete analytical characterization of long-run outcomes in an endowment economy populated by two classes of competitive agents (called, for simplicity, two agents) who differ in their beliefs about the distribution of the stochastic aggregate endowment that follows a geometric Brownian motion. In particular, optimistic (pessimistic) agents overstate (understate) the growth rate of aggregate endowment and, consequently, also the returns on claims to this endowment. Agents are endowed with identical preferences and trade in complete markets.

I show that in the class of recursive preferences, there exist broad ranges of empirically plausible values for preference parameters under which agents with less accurate beliefs prevail or even dominate the economy in terms of their wealth share and hence affect equilibrium dynamics in the long run. Perhaps most interestingly, agents with arbitrarily large belief distortions can coexist with rational agents in the long-run equilibrium under preference parameterizations typically estimated in asset pricing models, with risk aversion sufficiently higher than the inverse of IES; see, e.g., the long-run risk literature initiated by Bansal and Yaron (2004). In contrast to a large literature on market selection initiated by Alchian (1950) and Friedman (1953), belief heterogeneity should thus be viewed as a natural long-run outcome.

The paper expounds the market forces generating these results. The long-run wealth distribution in the economy is determined by relative *logarithmic* wealth growth rates of the two agents. Agents in an economy can accumulate wealth by choosing to hold portfolios with high expected *logarithmic* returns and by choosing a high saving rate.[1] The decoupling of risk aversion and IES effectively separates these two decisions. The portfolio choice is driven by the risk-return tradeoff that interacts

[1]To illuminate the implications of the difference between expected level and logarithmic returns for wealth accumulation, consider a simple case of a risk-neutral agent who starts with a given wealth of $k$ dollars, engages in a sequence of \$1 coin flip bets that win with probability 0.5, and quits when her wealth reaches zero. While the net expected *level* return is zero and the agent views these bets as fair and is willing to participate in them, she ultimately ends up with zero wealth with probability one. The expected *logarithmic* return on a sequence of these bets is negative and converges to minus infinity. This argument, based on the standard Jensen's inequality, is closely related to the literature on growth-optimal portfolios, initiated by Kelly (1956) and Breiman (1960, 1961).

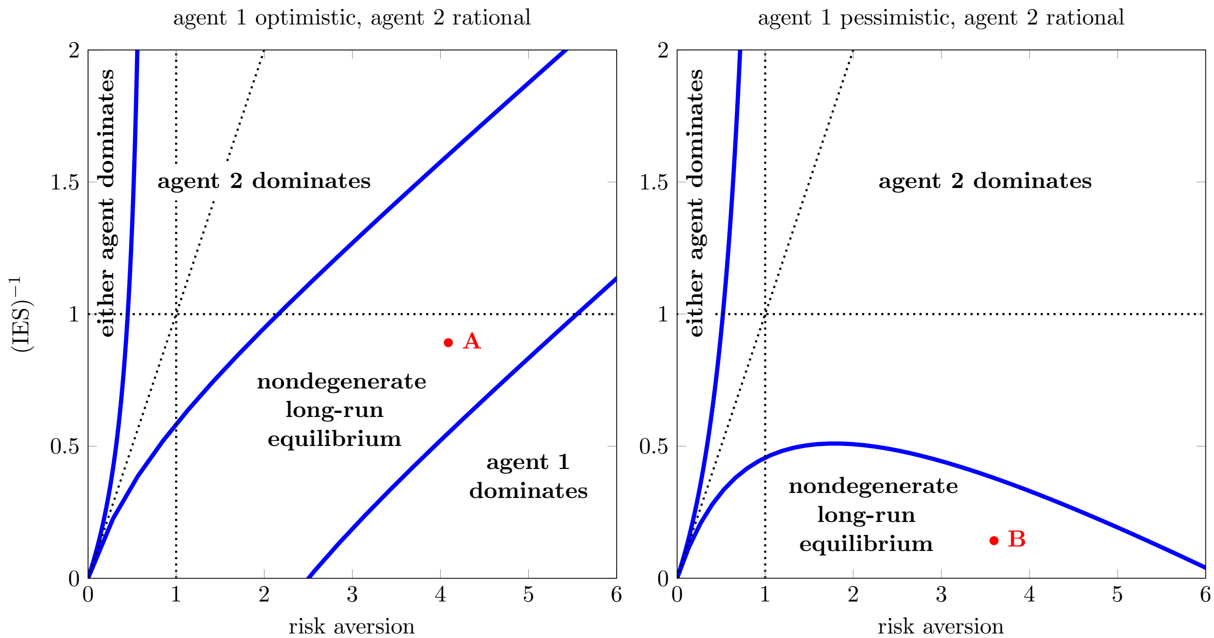


Figure 1: Survival regions for two pairs of agents' beliefs. **A** and **B** are two particular economies discussed in the text in which both agents survive.

risk aversion with subjective beliefs about expected asset returns, while the consumption-saving decision is determined through the interaction between IES and perceived expected returns on the agent's portfolio. I establish how the portfolio and saving mechanisms emerge from equilibrium price dynamics and determine novel long-run outcomes in the recursive preference setup.

The paper uncovers a crucial interaction between the use of risky assets for risk sharing and saving on the one hand and, on the other hand, as a speculative tool to trade on belief differences. Specifically, I identify three channels through which individual choices vis-à-vis equilibrium prices influence long-run wealth accumulation:

1. *Risk premium channel*: More optimistic agents hold larger positions in risky assets and thus benefit from high risk premia.

2. *Speculative volatility channel*: Speculative behavior arising from differences in beliefs makes agents choose portfolios with more volatile returns which lowers expected *logarithmic* returns.

3. *Saving channel*: Agents with a high perceived expected return on their portfolio choose a high (low) saving rate when IES is high (low) which aids (harms) their wealth growth.

Figure 1 provides an illustration of the results derived in the paper. Consider an economy where agent 2 has correct beliefs while agent 1 is optimistic (left panel) or pessimistic (right panel). Each panel shows long-run survival outcomes for economies with alternative combinations of preference parameters (which are always identical for both agents). Risk aversion is displayed on the horizontal axis, while the inverse of IES on the vertical axis. The dotted upward sloping line represents parameter combinations that correspond to CRRA preferences. The rational agent 2 always dominates in the neighborhood of the diagonal, which continuously extends existing survival results for separable preferences. Points **A** and **B** correspond to two economies in which both agents survive in the long run. Both economies feature high risk aversion relative to the inverse of IES.

High risk aversion in economy **A** implies that equilibrium risk premia are high. The *risk premium channel* then favors the more optimistic agent who holds a larger share of wealth invested in the risky asset, earns a higher expected return on her portfolio and accumulates wealth at a faster rate. But as the wealth share of the optimistic agent increases, the equilibrium risk premium declines, weakening the risk premium channel. Equilibrium asset price dynamics thus act as a balancing force, slowing down the rate of wealth accumulation of the optimistic agent. For a nontrivial set of moderately high values of the risk aversion parameter in Figure 1 that includes economy **A**, this mechanism preserves a nondegenerate wealth distribution in the long run in which an optimistic agent coexists with a rational agent despite her incorrect beliefs.

The *saving channel* aids survival of the optimistic agent in economy **A**, as well as of the pessimistic agent in economy **B**. When IES is above one, agent's saving rate is increasing in the subjective expected return on her portfolio. The saving channel then acts as a converging force that preserves long-run heterogeneity as long as the agent with a negligible wealth share (regardless of her identity) chooses a portfolio with a higher *subjective* expected level return than her large counterpart. This happens because equilibrium asset prices have to adjust to induce the large agent to hold the market portfolio so that markets clear, while the negligible agent at these prices forms a speculative portfolio with large positions in assets which she believes are underpriced. The negligible agent consequently chooses a high saving rate and in this way 'outsaves' her extinction.

The *speculative volatility channel*, on the other hand, acts as a diverging force on the wealth distribution, and determines long-run dynamics when risk aversion is low. Agents whose wealth share becomes negligible, again regardless of their identity, choose highly volatile speculative portfolios with low expected *logarithmic* returns, and are driven further to extinction.

I provide a complete analytical characterization of long-run outcomes for the whole parameter space and isolate the three channels described above. Several conclusions stand out.

First, the channels for the survival mechanism highlight the critical separate contributions of portfolio and consumption-saving decisions and their interaction with endogenously determined equilibrium price dynamics. In order for the two agents to coexist, equilibrium prices always have to be conducive to the survival of the negligible agent, and thus have to adjust when the wealth shares of the two agents switch. Recursive preferences play a crucial role in shaping these results.

Second, survival of agents with distorted beliefs is a robust outcome. Agents with incorrect beliefs can survive or dominate in bounded and unbounded economies populated by rational agents for a wide range of preference parameters. Moreover, these results do not hinge on belief distortions being small, or symmetric as in Scheinkman and Xiong (2003); in fact, as we will see, they hold for agents with arbitrarily large and arbitrarily asymmetric belief distortions.

Third, unlike in the separable utility case, long-run prospects of optimistic and pessimistic agents differ and do not depend solely on the magnitude of the belief distortions. This reflects the asymmetric effects of the above channels on optimists and pessimists.

Finally, equilibria in which agents with heterogeneous beliefs coexist in the long run occur for parameter combinations that are empirically relevant. In particular, risk aversion has to be sufficiently high to prevent the speculative volatility channel to dominate, and IES has to be sufficiently high to incentivize agents with a small wealth share to choose a high saving rate vis-à-vis the high subjective expected return on her portfolio, thus outsaving her extinction.

The paper also contributes to the literature along the methodological and technical dimension. First, I provide a novel rigorous proof of the existence and properties of the continuous-time optimal allocation problem with heterogeneous agents endowed with recursive preferences, formulated as a dynamic problem with stochastic Pareto weights. Second, I prove that this class of problems can be studied by focusing on the boundary behavior of the economy when one of the agents becomes negligible, which is often significantly simpler—in my case, I obtain a complete analytical characterization of the survival results despite the fact that the full model does not have a closed-form solution.

In this respect, the paper combines insights from the literature on long-run consumption dynamics in complete-market economies with recursive methods used to analyze allocations under non-separable preferences. Studies focusing on separable preferences (Sandroni (2000), Blume and Easley (2006), Yan (2008), Kogan, Ross, Wang, and Westerfield (2017) and others) provided very general conditions under which only agents with most accurate beliefs dominate the market in the long run. While optimal allocations in endowment economies with separable preferences can be solved for using a static planner's problem, non-separable preferences require the use of recursification methods based on Lucas and Stokey (1984) and Kan (1995). Specifically, the approach in my paper extends the continuous-time formulation in Dumas, Uppal, and Wang (2000). Anderson (2005), Mazoy (2005), Colacito, Croce, and Liu (2017) and others used these methods to study economies populated by agents with heterogeneous preferences. Bhandari (2015) and Guerdjikova and Sciubba (2015), specifically, focused on models with ambiguity averse consumers.

The framework studied in this paper is kept simple to yield analytical tractability but the key economic forces identified here hold more broadly. Baker, Hollifield, and Osambela (2016) and Pohl, Schmedders, and Wilms (2017) use these insights to construct models with subjective beliefs and a non-degenerate long-run distribution of wealth that feature a production side and long-run risks, respectively. Dindo (2015) confirms particular analytical results in a discrete-time environment. Since the methodological approach used in my paper differs significantly from much of the survival literature, I defer a more detailed discussion and comparison to the literature to Section 6.

The rest of the paper is organized as follows. Section 2 outlines the environment and derives the planner's problem. The proof of the existence and uniqueness of the solution is deferred to Online Appendix OA.9.[2] Sections 3 and 4 present the survival results in the form of tight analytical conditions for survival and extinction, followed by a discussion of asset price implications in Section 5. Section 6 revisits the methodological contribution vis-à-vis the existing literature and Section 7 concludes. The Appendix contains further proofs omitted from the main text. Additional material that provides more detail and extends the analysis is available in the Online Appendix.

[2] https://www.borovicka.org/files/research/survival_heterogeneous_beliefs_online_appendix.pdf

## 2 Optimal allocations under heterogeneous beliefs

I analyze the dynamics of equilibrium allocations in a continuous-time endowment economy populated by two types of infinitely-lived agents endowed with identical recursive preferences. I call an economy where both agents have strictly positive wealth shares a heterogeneous economy. A homogeneous economy is populated by a single agent only. The term 'agent' refers to an infinitesimal competitive representative of the particular type.

Agents differ in their subjective beliefs about the distribution of future quantities but are firm believers in their probability models and 'agree to disagree' about their beliefs as in Morris (1995). Since they do not interpret their belief differences as a result of information asymmetries, there is no strategic trading behavior.

Without introducing any specific market structure, I assume that markets are dynamically complete in the sense of Harrison and Kreps (1979). This allows me to sidestep the problem of directly calculating the equilibrium by considering a planner's problem. The discussion of market survival then amounts to the analysis of the dynamics of Pareto weights associated with this planner's problem (Negishi (1960)). Optimal allocations and continuation values generate a valid stochastic discount factor and a replicating trading strategy for the decentralized equilibrium.

In this section, I specify agents' preferences and belief distortions, and lay out the planner's problem. I utilize the framework introduced by Dumas, Uppal, and Wang (2000), and exploit the observation that belief heterogeneity can be analyzed in their framework without increasing the degree of complexity of the problem. The method then leads to a Hamilton–Jacobi–Bellman equation for the planner's value function. An analogous problem formulated in discrete time is available in the Online Appendix, Section OA.8.

### 2.1 Information structure and beliefs

The stochastic structure of the economy is given by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with an augmented filtration defined by a family of $\sigma$-algebras $\{\mathcal{F}_t\},\ t \geq 0$ generated by a univariate Brownian motion $W$.[3] The scalar aggregate endowment process $Y$ follows a geometric Brownian motion

$$d \log Y_t = \mu_y dt + \sigma_y dW_t, \qquad Y_0 > 0 \tag{1}$$

with given parameters $\mu_y$ and $\sigma_y$. Agents of type $n \in \{1, 2\}$ are endowed with identical preferences but differ in the subjective probability measures they use to assign probabilities to future events. They agree on $\mu_y$ and $\sigma_y$, and observe realizations of $W_t$ and $Y_t$ but disagree about their distribution. I model the belief distortion of agent $n$ using an adapted process $u^n$ such that the process

$$M_t^n \doteq \left( \frac{dQ^n}{dP} \right)_t = \exp\left( -\frac{1}{2} \int_0^t |u_s^n|^2 ds + \int_0^t u_s^n dW_s \right), \tag{2}$$

[3] Given the continuous-time nature of the problem, equalities are meant in the appropriate almost-sure sense. I also assume that all processes, in particular belief distortions, individual endowments and permissible trading strategies, are adapted to $\{\mathcal{F}_t\}$ and satisfy usual regularity conditions like square integrability over finite horizons, so that stochastic integrals are well defined and pathological cases are avoided (see, e.g., Huang and Pagès (1992)). Under the parameter restrictions below, constructed equilibria satisfy these assumptions.

is a martingale under $P$. The martingale $M^n$ is called the Radon–Nikodým derivative or the belief ratio and defines the subjective probability measure $Q^n$ that characterizes the beliefs of agent $n$. The Radon–Nikodým derivative measures the disparity between the subjective and true probability measures.

Subjective beliefs are constructed so that the agents agree with the data generating measure on zero-probability finite-horizon events.[4] While a likelihood evaluation of past observed data reveals that the view of an agent with distorted beliefs becomes less and less likely to be correct as time passes, absolute continuity of the measure $Q^n$ with respect to $P$ over finite horizons implies that she cannot refute her view of the world as impossible in finite time.

From now on, I assume that both agents have constant belief distortions $u^n$. These belief distortions have a clear economic interpretation. The Girsanov theorem implies that agent $n$, whose deviation from rational beliefs is described by $u^n$, views the evolution of the Brownian motion $W$ as distorted by a drift component $u^n$, i.e., $dW_t = u^n dt + dW_t^n$, where $W^n$ is a Brownian motion under $Q^n$. Consequently, the aggregate endowment is perceived to contain an additional drift component $u^n \sigma_y$, and $u^n$ can be interpreted as a degree of optimism or pessimism about the distribution of future aggregate endowment. Conditional on time 0, agent $n$ believes that the aggregate endowment $Y$ follows

$$d \log Y_t = (\mu_y + u^n \sigma_y)\, dt + \sigma_y dW_t^n,$$

i.e., that the distribution of $\log Y_t$ conditional on $\mathcal{F}_0$ is Normal with mean $\log Y_0 + (\mu_y + u^n \sigma_y)\, t$ and variance $\sigma_y^2 t$. When $\sigma_y = 0$, the distinction between optimism and pessimism loses its meaning but the survival problem is still nondegenerate when agents contract upon the realizations of the process $W$.

## 2.2 Recursive utility

Agents endowed with separable preferences reduce intertemporal compound lotteries (different payoff streams allocated over time) to atemporal simple lotteries that resolve uncertainty at a single point in time. In the Arrow–Debreu world with separable preferences, once trading of state-contingent securities for all future periods is completed at time 0, uncertainty about the realized path of the economy can be resolved immediately without any consequences for the ex-ante preference ranking of the outcomes by the agents.

Kreps and Porteus (1978) relaxed the separability assumption by axiomatizing discrete-time preferences where temporal resolution of uncertainty matters and preferences are not separable. While intratemporal lotteries in the Kreps–Porteus axiomatization still satisfy the von Neumann–

[4]In order for the belief heterogeneity not to vanish in the long run, the measures $P$ and $Q^n$ cannot be mutually absolutely continuous. Sandroni (2000) and Blume and Easley (2006) link absolute continuity of the subjective probability measures to merging of agents' beliefs. However, given the construction of $M^n$, the restrictions of the measures $P$ and $Q^n$, $n \in \{1, 2\}$ to $\mathcal{F}_t$ for every $t \geq 0$ are equivalent (e.g., Revuz and Yor (1999), Section VIII). The construction prevents arbitrage opportunities in finite-horizon strategies, and the Pareto optimal allocation can thus be decentralized using dynamic trading. The martingale representation theorem (e.g., Øksendal (2007), Theorem 4.3.4) implies that modeling belief distortions under Brownian information structures using martingales of the form (2) is essentially without loss of generality. Online Appendix, Section OA.3, provides further details.

Morgenstern expected utility axioms, intertemporal lotteries cannot in general be reduced to atemporal ones. The work by Epstein and Zin (1989, 1991) extended the results of Kreps and Porteus (1978), and initiated the widespread use of recursive preferences in the asset pricing literature. Duffie and Epstein (1992a,b) formulated the continuous-time counterpart of the recursion.[5]

I utilize a characterization based on the more general variational utility approach studied by Geoffard (1996) in the deterministic case and El Karoui, Peng, and Quenez (1997) in a stochastic environment.[6] They show that recursive preferences can be represented as a solution to the maximization problem

$$\lambda_t^n V_t^n (C^n) = \sup_{\nu^n} E_t^{Q^n} \left[ \int_t^\infty \lambda_s^n F\left(C_s^n, \nu_s^n\right) ds \right] \tag{3}$$

subject to

$$d \log \lambda_t^n = -\nu_t^n dt, \qquad \lambda_0^n > 0, \quad t \geq 0. \tag{4}$$

where $\nu^n$ is called the discount rate process, and $\lambda^n$ the discount factor process. The felicity function $F(C, \nu)$ encodes the contribution of the consumption stream $C$ to present utility. This representation closely links recursive preferences to the literature on endogenous discounting, initiated by Koopmans (1960) and Uzawa (1968).

For the case of the Duffie–Epstein–Zin preferences, the felicity function is given by

$$F(C, \nu) = \beta \frac{C^{1-\gamma}}{1-\gamma} \left( \frac{1 - \gamma - (1-\rho) \frac{\nu}{\beta}}{\rho - \gamma} \right)^{\frac{\gamma - \rho}{1-\rho}}, \tag{5}$$

with parameters satisfying $\gamma, \rho, \beta > 0$. Preferences specified by this felicity function[7] are homothetic and exhibit a constant relative risk aversion with respect to intratemporal wealth gambles $\gamma$ and (under intratemporal certainty) a constant intertemporal elasticity of substitution $\rho^{-1}$. Parameter $\beta$ is the time preference coefficient. In the case when $\gamma = \rho$, the utility reduces to the separable CRRA utility with the coefficient of relative risk aversion $\gamma$.

Formula (3), together with an application of the Girsanov theorem, suggests that it is advantageous to combine the contribution of the discount factor process $\lambda^n$ and the martingale $M^n$ that specifies the belief distortion in (2):

**Definition 2.1** *A modified discount factor process $\bar{\lambda}^n$ is a discount factor process that incorporates the martingale $M^n$ arising from the belief distortion, $\bar{\lambda}^n \doteq \lambda^n M^n$.*

[5] Duffie and Epstein (1992b) provide sufficient conditions for the existence of the recursive utility process for the infinite-horizon case but these are too strict for the preference specification utilized in this paper. Similarly, the results from Duffie and Lions (1992) for Markov environments do not apply for all cases considered here. Schroder and Skiadas (1999) establish conditions under which the continuation value is concave, and provide further technical details. Skiadas (1997) shows a representation theorem for the discrete-time version of recursive preferences with subjective beliefs.

[6] Hansen (2004) offers a tractable summary of the link between recursive and variational utility. Interested readers may refer to the Online Appendix, Section OA.2, for a more detailed discussion.

[7] The cases of $\rho \to 1$ and $\gamma \to 1$ can be obtained as appropriate limits. The maximization problem (3) assumes that the felicity function is concave in its second argument. When it is convex, the formulation becomes a minimization problem.

Applying Itô's lemma to $\bar{\lambda}^n$ leads to a maximization problem under the true probability measure

$$\bar{\lambda}_t^n V_t^n (C^n) = \sup_{\nu^n} E_t \left[ \int_t^\infty \bar{\lambda}_s^n F (C_s^n, \nu_s^n) \, ds \right] \tag{6}$$

subject to

$$d \log \bar{\lambda}_t^n = - \left( \nu_t^n + \frac{1}{2} (u^n)^2 \right) dt + u^n dW_t, \qquad \bar{\lambda}_0^n > 0, \quad t \geq 0. \tag{7}$$

Problem (6)–(7) indicates that $F(C_t^n, \nu_t^n)$ can be viewed as a generalization of the period utility function with a potentially stochastic rate of time preference $\nu_t^n$ that depends on the properties of the consumption process and thus arises endogenously in a market equilibrium. Moreover, belief distortions are now fully incorporated in the framework of Dumas, Uppal, and Wang (2000)—the only difference is that the modified discount factor process is not locally predictable. The term $-\frac{1}{2}(u^n)^2$ in the drift of $\log \bar{\lambda}_t^n$ reflects the average bias arising from evaluating the utility flow under the subjective belief.

The diffusion term $u^n dW_t$ has an intuitive interpretation. Consider an optimistic agent with $u^n > 0$. This agent's beliefs are distorted in that the mass of the distribution of $dW_t$ is shifted to the right—the agent effectively overweighs good realizations of $dW_s$. Formula (7) indicates that under the true probability measure, positive realizations of $dW_t$ increase $\bar{\lambda}_t^n$, which implies that the optimistic agent discounts positive realizations of $dW_t$ less than negative ones.

### 2.3 Planner's problem and optimal allocations

I follow Dumas, Uppal, and Wang (2000) and introduce a fictitious planner who maximizes a weighted average of the continuation values of the two agents.[8] The pair of strictly positive initial Pareto weights $\bar{\lambda}_0 \doteq (\bar{\lambda}_0^1, \bar{\lambda}_0^2)$ determines the initial distribution of wealth. The problem of an individual agent (6)–(7) is homogeneous degree one in the modified discount factors and homogeneous degree $1-\gamma$ in consumption, and this property carries over to the planner's problem. Define the consumption *shares* of the two agents as $\zeta^n \doteq C^n / Y$, $n \in \{1, 2\}$.

**Definition 2.2** *The planner's* value function *is the solution to the problem*

$$J(\bar{\lambda}_0, Y_0) \doteq \sup_{(C^1, C^2)} \sum_{n=1}^{n} \bar{\lambda}_0^n V_0^n (C^n) = \sup_{(\zeta^1, \zeta^2, \nu^1, \nu^2)} \sum_{n=1}^{2} E_0 \left( \int_0^\infty \bar{\lambda}_t^n Y_t^{1-\gamma} F(\zeta_t^n, \nu_t^n) \, dt \right) \tag{8}$$

*subject to the law of motion for the modified discount factors (7) with initial conditions $(\bar{\lambda}_0^1, \bar{\lambda}_0^2)$, and the feasibility constraint $\zeta^1 + \zeta^2 \leq 1$.*

The planner's problem is well-defined under a simple restriction on the parameters of the economy, imposed in Assumption A.1 in Appendix A, which is maintained throughout the text. The restriction effectively states that the agents have to be sufficiently impatient ($\beta$ is sufficiently high)

[8]The validity of this approach for a finite-horizon economy is discussed in Dumas, Uppal, and Wang (2000) and Schroder and Skiadas (1999). The infinite-horizon problem in (8) is a straightforward extension when individual continuation values are well-defined.

for the equilibrium to exist. Since the survival results do not depend on $\beta$, Assumption A.1 is not material for the economic substance of the problem. See Appendix A for details.

The planner's problem (8) suggests that we can interpret the modified discount factor processes $\bar{\lambda}^n$ as stochastic Pareto weights. Dumas, Uppal, and Wang (2000) show that in a Markov environment without belief heterogeneity, the discount factor processes $\lambda^n$ in (3)–(4) serve as new state variables that allow a recursive formulation of the problem using the Hamilton–Jacobi–Bellman (HJB) equation. The same conclusion holds under belief heterogeneity for the modified discount factor processes $\bar{\lambda}^n$, since the introduction of belief heterogeneity kept the structure of the problem unchanged. Indeed, if $\bar{\lambda}^n_0$ are initial weights, then $\bar{\lambda}^n_t$ are the consistent state-dependent weights for the continuation problem of the planner at time $t$.

### 2.4 Hamilton–Jacobi–Bellman equation

The planner's problem has an appealing Markov structure. The value function (8) for the planner's problem at time $t$ is homogeneous degree 1 in $\bar{\lambda} = \left(\bar{\lambda}^1, \bar{\lambda}^2\right)$, homogeneous degree $1-\gamma$ in $Y$ and can be written as

$$J\left(\bar{\lambda}_t, Y_t\right) = \left(\bar{\lambda}^1_t + \bar{\lambda}^2_t\right) Y_t^{1-\gamma} \widetilde{J}\left(\theta_t\right)$$

where $\theta \doteq \bar{\lambda}^1 / \left(\bar{\lambda}^1 + \bar{\lambda}^2\right)$ represents the Pareto share of agent 1 and acts as the only relevant state variable in the problem. The planner's problem can thus be characterized as a solution to a Hamilton–Jacobi–Bellman equation for $\widetilde{J}(\theta)$. Obviously, $\theta$ is bounded between zero and one. It will become clear that for strictly positive initial weights, the boundaries are unattainable, so that $\theta$ evolves on the open interval $(0,1)$. The proof of the following proposition together with further properties of the value function including its homogeneity and technical discussion is in Online Appendix OA.9.

**Proposition 2.3** *The Hamilton–Jacobi–Bellman equation*

$$\begin{aligned} 0 = \sup_{(\zeta^1,\zeta^2,\nu^1,\nu^2)} & \theta F\left(\zeta^1,\nu^1\right) + (1-\theta) F\left(\zeta^2,\nu^2\right) + \\ & + \left[-\theta\nu^1 - (1-\theta)\nu^2 + \left(\theta u^1 + (1-\theta) u^2\right)(1-\gamma)\sigma_y + (1-\gamma)\mu_y + \frac{1}{2}(1-\gamma)^2\sigma_y^2\right]\widetilde{J}(\theta) \\ & + \theta(1-\theta)\left[\nu^2 - \nu^1 + \left(u^1 - u^2\right)(1-\gamma)\sigma_y\right]\widetilde{J}_\theta(\theta) + \frac{1}{2}\theta^2(1-\theta)^2\left(u^1-u^2\right)^2\widetilde{J}_{\theta\theta}(\theta) \end{aligned} \tag{9}$$

*with boundary conditions $\widetilde{J}(0) = \bar{V}^2$ and $\widetilde{J}(1) = \bar{V}^1$ has a unique bounded twice continuously differentiable solution such that $J\left(\bar{\lambda}_t, Y_t\right) = \left(\bar{\lambda}^1_t + \bar{\lambda}^2_t\right) Y_t^{1-\gamma}\widetilde{J}\left(\theta_t\right)$ is the planner's value function.*

The dynamics of $\theta$ are central to the study of survival in this paper. They dictate how the planner adjusts the weights of the two agents, and thus their current consumption and wealth, over time. In this respect, the only relevant force for survival is the willingness of the planner to increase the Pareto weight of the agent that becomes negligible and faces the risk of becoming extinct. Hence only the boundary behavior of the scalar Itô process $\theta$ matters. Despite the fact that equation (9) generally does not have a closed-form solution for $\widetilde{J}(\theta)$, this boundary behavior can be characterized analytically by studying the limiting behavior of the objective function.[9]

# 3 Long-run wealth distribution and survival

In this section, I formalize the exact relationship between survival and the boundary behavior of the Pareto share $\theta$ (Proposition 3.2), link it to the equilibrium dynamics of the wealth distribution (Proposition 3.3) and derive analytical formulas for the wealth dynamics at the boundaries (Proposition 3.4). These results provide a complete analytical characterization of survival outcomes in terms of fundamental parameters of the economy, and reveal the contribution of two crucial equilibrium forces—returns on agents' portfolios, and agents' saving rates.

Specifically, I rely on ergodic properties of $\theta$ to investigate the existence of its unique stationary distribution. The derived sufficient conditions depend on the behavior of the difference of agents' endogenous discount rates. In a decentralized economy, these *relative patience* conditions can be reinterpreted in terms of the difference in expected logarithmic growth rates of individual wealth.

Since the analyzed model includes growing and decaying economies, I am interested in a measure of *relative* survival, characterized by the behavior of the Pareto *share* $\theta$. The following definition distinguishes between survival along individual paths and almost-sure survival.

**Definition 3.1** *Agent 1 becomes extinct along the path $\omega \in \Omega$ if $\lim_{t\to\infty} \theta_t(\omega) = 0$. Otherwise, agent 1 survives along the path $\omega$. Agent 1 dominates in the long run along the path $\omega$ if $\lim_{t\to\infty} \theta_t(\omega) = 1$.*

*Agent 1 becomes extinct (under measure $P$) if $\lim_{t\to\infty} \theta_t = 0$, $P$-a.s. Agent 1 survives if $\limsup_{t\to\infty} \theta_t > 0$, $P$-a.s. Agent 1 dominates in the long run if $\lim_{t\to\infty} \theta_t = 1$, $P$-a.s. Agent 1 dominates with a strictly positive probability if $P(\lim_{t\to\infty} \theta_t = 1) > 0$.*

Yan (2008) or Kogan, Ross, Wang, and Westerfield (2017) use asymptotic behavior of consumption shares $\zeta^n$ to define a measure of survival. Since the consumption share is continuous and strictly increasing in the Pareto share $\theta$ and the limits are $\lim_{\theta\searrow 0} \zeta^1(\theta) = 0$ and $\lim_{\theta\nearrow 1} \zeta^1(\theta) = 1$ (see Remark OA.1 in the Online Appendix), the two measures are equivalent in this setting.

## 3.1 Long-run distribution of the Pareto share

Recall the dynamics of the modified discount factor processes $\bar{\lambda}^n$ in (7). To state the survival results, it is convenient to consider the transformation $\vartheta \doteq \log(\theta/(1-\theta)) = \log\left(\bar{\lambda}^1/\bar{\lambda}^2\right)$. Then

$$d\vartheta_t = \left[\left(\nu_t^2 + \frac{1}{2}\left(u^2\right)^2\right) - \left(\nu_t^1 + \frac{1}{2}\left(u^1\right)^2\right)\right] dt + \left(u^1 - u^2\right) dW_t \doteq \mu_{\vartheta,t} dt + \sigma_{\vartheta,t} dW_t. \tag{10}$$

[9] Equation (9) is not specific to the planner's problem (8). For instance, Gârleanu and Panageas (2015) use the martingale approach to directly analyze the equilibrium in an economy with agents endowed with heterogeneous recursive preferences, and show that they can derive their asset pricing formulas in closed form up to the solution of a nonlinear ODE that has the same structure as (9), which they have to solve for numerically. The analytical characterization of the boundary behavior of the ODE derived in this paper is thus applicable to a wider class of recursive utility models, and can aid numerical calculations which are often unstable in the neighborhood of the boundaries in this type of problems.

Under nonseparable preferences, the discount rates $\nu_t^n = \nu^n(\theta_t)$ are determined endogenously in the model as part of the solution to problem (9). Intuitively, one would expect a stationary distribution for $\theta$ to exist if the process exhibits sufficient pull toward the center of the interval when close to the boundaries. This is formalized in the following conditions on the drift coefficient $\mu_{\vartheta,t} = \mu_\vartheta(\theta_t)$:

**Proposition 3.2** *Define the following 'repelling' conditions (i) and (ii), and their 'attracting' counterparts (i') and (ii'):*

$$\textbf{(i)} \quad \lim_{\theta \searrow 0} \mu_\vartheta(\theta) > 0 \qquad \textbf{(i}'\textbf{)} \quad \lim_{\theta \searrow 0} \mu_\vartheta(\theta) < 0$$

$$\textbf{(ii)} \quad \lim_{\theta \nearrow 1} \mu_\vartheta(\theta) < 0 \qquad \textbf{(ii}'\textbf{)} \quad \lim_{\theta \nearrow 1} \mu_\vartheta(\theta) > 0$$

*Then the following statements are true:*

**(a)** *If conditions (i) and (ii) hold, then both agents survive under $P$.*

**(b)** *If conditions (i) and (ii') hold, then agent 1 dominates in the long run under $P$*

**(c)** *If conditions (i') and (ii) hold, then agent 2 dominates in the long run under $P$.*

**(d)** *If conditions (i') and (ii') hold, then there exist sets $S^1, S^2 \subset \Omega$ which satisfy*

$$S^1 \cap S^2 = \emptyset, \quad P\left(S^1\right) \neq 0 \neq P\left(S^2\right), \quad \textit{and} \quad P\left(S^1 \cup S^2\right) = 1$$

*such that agent 1 dominates in the long run along each path $\omega \in S^1$ and agent 2 dominates in the long run along each path $\omega \in S^2$.*

*The conditions are also the least tight bounds of this type.*

Given the dynamics of the transformed Pareto share (10), conditions (i) and (ii) are jointly sufficient for the existence of a unique stationary density $q(\theta)$. The proof of Proposition 3.2 is based on the Feller (1952, 1957) classification of boundary behavior of diffusion processes, discussed in Karlin and Taylor (1981). The four 'attracting' and 'repelling' conditions are only sufficient and their combinations stated in Proposition 3.2 are not exhaustive. However, the only unresolved cases are knife-edge cases involving equalities in the conditions of the Proposition, which are only of limited importance in the analysis below.

I call the difference in the discount rates $\nu^2(\theta) - \nu^1(\theta)$ *relative patience* because it captures the difference in discounting of future felicity in the variational utility specification (3) between the two agents. Conditions in Proposition 3.2 have an intuitive interpretation. Survival condition (i) states that agent 1 survives under the true probability measure even in cases when her beliefs are more distorted, $\left|u^1\right| > \left|u^2\right|$, as long as her relative patience becomes sufficiently high to overcome the distortion when her Pareto share vanishes. The discount rate $\nu^n$ encodes not only a pure time preference but also the interaction of current discounting with the dynamics of the continuation values that reflects the behavior of the equilibrium consumption streams.[10]

[10] Lucas and Stokey (1984) impose a similar condition called *increasing marginal impatience*, see Section 6.

### 3.2 Decentralization and equilibrium wealth dynamics

Proposition 3.2 states the survival conditions in terms of the endogenous discount rates $\nu^n$. Now I link these conditions to the equilibrium wealth dynamics in the economy, and evaluate analytically the regions in the parameter space in which these conditions hold.

The proof strategy in this section relies on a decentralization argument and utilizes the asymptotic properties of the differential equation (9) for the planner's continuation value. The economy is driven by a single Brownian shock, and two suitably chosen assets that can be continuously traded are therefore sufficient to complete the markets in the sense of Harrison and Kreps (1979). Let the two traded assets be an infinitesimal risk-free bond in zero net supply that yields a risk-free rate $r_t = r(\theta_t)$ and a claim on the aggregate endowment with price $A_t = Y_t \xi(\theta_t)$, where $\xi(\theta)$ is the aggregate wealth-consumption ratio. Individual wealth levels are denoted $A_t^n = Y_t \zeta^n(\theta_t)\, \xi^n(\theta_t)$, where $\xi^n(\theta)$ are the individual wealth-consumption ratios. Individual wealth levels follow the law of motion $d \log A_t^n = \mu_{A^n}(\theta_t)\, dt + \sigma_{A^n}(\theta_t)\, dW_t$.

The results reveal that as the Pareto share of one of the agents converges to zero, the infinitesimal returns associated with the two assets converge to those which prevail in a homogeneous economy populated by the agent with the large Pareto share. This feature is closely related to the *price impact* that vanishing agents have on equilibrium asset prices, and I discuss the equilibrium asset price dynamics in detail in Section 5.

**Proposition 3.3** *The boundary behavior of Pareto shares and agents' wealth satisfies*

$$\lim_{\theta \to \bar{\theta}} \gamma^{-1} \mu_{\vartheta}(\theta) = \lim_{\theta \to \bar{\theta}} \left[ \mu_{A^1}(\theta) - \mu_{A^2}(\theta) \right], \qquad \bar{\theta} \in \{0, 1\}.$$

*Survival conditions in Proposition 3.2 can thus be expressed in terms of relative wealth dynamics of the two agents.*

Verifying the conditions in Proposition 3.2 therefore amounts to checking that the expected growth rate of the logarithm of wealth is higher for the agent who is at the brink of extinction. The two central forces underlying wealth accumulation and long-run survival are agents' portfolio allocation and consumption-saving decisions. The rate of wealth accumulation can therefore be decomposed into the return on the agent's portfolio net of the consumption rate:

$$d \log A_t^n = d \log R_t^n - \left(\xi_t^n\right)^{-1} dt.$$

Both terms on the right-hand side can be characterized analytically at the boundaries. Denoting $d \log R_t^n = \mu_{R^n}(\theta_t)\, dt + \sigma_{R^n}(\theta_t)\, dW_t$ where $\mu_{R^n}(\theta_t)$ is the expected logarithmic return on agent's $n$ portfolio (and $\mu_{R^n}(\theta_t) + \frac{1}{2}\sigma_{R^n}^2(\theta_t)$ the expected level return), we can establish the following decomposition for the case when agent 1 becomes negligible ($\theta \searrow 0$). The case $\theta \nearrow 1$ is symmetric.

**Proposition 3.4** *As $\theta \searrow 0$, the difference in the logarithmic wealth growth rates between the agent with negligible wealth and the large agent can be written as*

$$\lim_{\theta \searrow 0} \left[ \mu_{A^1}(\theta) - \mu_{A^2}(\theta) \right] = \lim_{\theta \searrow 0} \left[ \mu_{R^1}(\theta) - \mu_{R^2}(\theta) \right] + \lim_{\theta \searrow 0} \left[ \left(\xi^2(\theta)\right)^{-1} - \left(\xi^1(\theta)\right)^{-1} \right]$$

*where the difference in the expected logarithmic portfolio returns is*

$$\lim_{\theta \searrow 0} \left[ \mu_{R^1}(\theta) - \mu_{R^2}(\theta) \right] = \underbrace{\frac{u^1 - u^2}{\gamma \sigma_y}}_{\substack{\text{difference in} \\ \text{portfolios}}} \underbrace{\left[ \gamma \sigma_y^2 - u^2 \sigma_y \right]}_{\text{risk premium}} - \underbrace{\frac{u^1 - u^2}{\gamma} \left( \sigma_y + \frac{1}{2} \frac{u^1 - u^2}{\gamma} \right)}_{\text{volatility penalty}} \tag{11}$$

*and the difference in consumption rates is given by*

$$\lim_{\theta \searrow 0} \left[ \left( \xi^2(\theta) \right)^{-1} - \left( \xi^1(\theta) \right)^{-1} \right] = \frac{1}{2} \frac{1-\rho}{\rho} \underbrace{\left[ 2\left( u^1 - u^2 \right) \sigma_y + \frac{\left( u^1 - u^2 \right)^2}{\gamma} \right]}_{\text{difference in subjective expected returns}}. \tag{12}$$

The proposition reveals a clear separation of the role of risk aversion and IES. The difference in the expected logarithmic portfolio returns at the boundary only depends on the relative risk aversion $\gamma$, not on the parameter $\rho$ that determines the IES. The first term represents the *risk premium channel*—the risk premium on the claim on aggregate consumption times the difference in the portfolio shares invested in the risky asset, obtained in equation (16). The risk premium itself is composed of the standard rational expectations premium $\gamma \sigma_y^2$ and a 'mispricing' effect $-u^2 \sigma_y$ (when the large agent 2 is optimistic, she overprices the risky asset which leads to a lower expected return). Since survival is driven by the expected *logarithmic* return, volatile portfolios are penalized by a lognormal correction, reflecting the *speculative volatility channel.* This volatility penalty is the dominant force for survival when risk aversion declines to zero ($\gamma \searrow 0$).

The difference in consumption rates consists of two components. The term in brackets is the difference between the expected portfolio return of agent 1 as perceived by agent 1, and the portfolio return of agent 2 as perceived by agent 2,

$$\left[ \mu_{R^1}(\theta_t) + \frac{1}{2} \sigma_{R^1}^2(\theta_t) + u^1 \sigma_{R^1}(\theta_t) \right] - \left[ \mu_{R^2}(\theta_t) + \frac{1}{2} \sigma_{R^2}^2(\theta_t) + u^2 \sigma_{R^2}(\theta_t) \right].$$

Here, $\mu_{R^n}(\theta_t) + \frac{1}{2} \sigma_{R^n}^2(\theta_t)$ is the objective expected level return on agent's $n$ portfolio, and $u^n \sigma_{R^n}(\theta_t)$ is the subjective bias. It is the *subjective* expected returns (computed under $Q^n$, not $P$) that enter the formula because the consumption-saving decision of the agent depends on the expected portfolio return as perceived by herself.

When IES $\rho^{-1} = 1$, the consumption-wealth ratios of the two agents are identical and equal to $\beta$ as in the case of myopic logarithmic utility, and the consumption-saving decision plays no role in the survival outcomes. When preferences are elastic ($\rho^{-1} > 1$), the saving rate is an increasing function of the subjective expected portfolio return and the difference in consumption rates is therefore negatively related to the difference in subjective expected returns—vis-à-vis a high expected return, the agent with elastic preferences decides to postpone consumption and tilt the consumption profile toward the future. This helps the agent with the higher expected subjective return outsave her extinction, reflecting the *saving channel* of survival.

### 3.3 Dependence of survival results on individual parameters

The results from Proposition 3.4 reveal that the survival regions depend on the ratios of parameters $u^1/\sigma_y$ and $u^2/\sigma_y$, and not on the three parameters independently. This is an important insight which shows that what matters for survival in this economy is the importance of aggregate fundamental risk embedded in $\sigma_y$ relative to the willingness of the agents to speculate, reflected in the magnitude of the belief distortions $u^n$. Large belief distortions encourage larger speculative portfolio positions with volatile returns and increase the role of the volatility penalty. Aggregate risk, on the other hand, discourages additional risk taking through speculation.

For example, if agent 2 has correct beliefs, $u^2 = 0$, the long-run survival outcome is the same whether we fix the belief distortion $u^1$ and make aggregate endowment deterministic, $\sigma_y \searrow 0$, or if we fix $\sigma_y$ and make the beliefs of agent 1 infinitely incorrect. I revisit these aspects of the survival results in the next section.

The survival results also do not depend on the time preference parameter $\beta$ and the growth rate of the economy $\mu_y$. Both these parameters affect individual consumption-saving decisions symmetrically, and hence they have no impact on the difference in the rates of wealth accumulation. This would no longer be true if, for instance, agents differed in the IES parameter.

In Section 4.4.3, I combine these insights and study an economy with constant aggregate endowment ($\mu_y = \sigma_y = 0$) to show that the survival results are not affected by the nonstationarity of aggregate endowment in a growing or decaying economy.

## 4 Survival regions

This section analyzes the regions of the parameter space in which agents with distorted beliefs survive or dominate the economy. It turns out that all four combinations generated by the pair of inequalities in Proposition 3.2 do occur in nontrivial parts of the parameter space.

Survival conditions in Proposition 3.4 depend only on parameters $\left(\gamma, \rho, u^1/\sigma_y, u^2/\sigma_y\right)$. Figure 2 provides a systematic treatment of the parameter space. Each panel plots the survival results in the 'risk aversion / inverse of IES' plane $(\gamma, \rho)$ for different levels of belief distortions. To keep the discussion focused, I concentrate on the case when agent 2 has correct beliefs, $u^2 = 0$. The Online Appendix considers additional cases when beliefs of both agents are distorted but these are all special cases of Proposition 3.2. To get an idea about the magnitude of the belief distortions, recall that an agent with $u^1 = 0.1$ distorts the annual growth rate of aggregate endowment by $u^1\sigma_y$, e.g., believes it to be 2.2% instead of 2% when $\sigma_y = 0.02$.

The shaded area represents the parameter combinations for which a nondegenerate long-run equilibrium exists. The blue dashed lines in the graphs depict parameter combinations for which condition **(i)** in Proposition 3.2 holds with equality (as $\theta \searrow 0$), while the solid red lines capture the same situation for condition **(ii)** (as $\theta \nearrow 1$). The results do not reveal what happens at these boundaries but the long-run outcomes for the interiors of the individual regions are completely characterized by the conditions in Proposition 3.2. The existing literature established that along the dotted diagonal, which represents the parameter combinations for separable CRRA preferences, the agent with more accurate beliefs (i.e., with a smaller $|u^n|$, in our case agent 2) dominates the economy in the long run.

It is useful to start by describing the asymptotic results as either risk aversion or intertemporal elasticity of substitution moves toward extreme values, holding other parameters fixed. These limiting cases isolate the role of the individual survival channels outlined in the introduction. Section 4.4 then analyzes the interaction of these forces in the whole parameter space.

**Corollary 4.1** *Holding other parameters fixed, for any given pair of beliefs $u^n$, $n \in \{1, 2\}$ and $\sigma_y > 0$, the survival restrictions imply the following asymptotic results under $P$.*

**(a)** *As agents become risk neutral ($\gamma \searrow 0$), each agent dominates in the long run with a strictly positive probability.*

**(b)** *As risk aversion increases ($\gamma \nearrow \infty$), the agent who is relatively more optimistic about the growth rate of aggregate endowment always dominates in the long run.*

**(c)** *As IES increases ($\rho \searrow 0$), the relatively more optimistic agent always survives. The relatively more pessimistic agent survives (and thus a nondegenerate long-run equilibrium exists) when risk aversion is sufficiently small.*

**(d)** *As IES decreases to zero ($\rho \nearrow \infty$), a nondegenerate long-run equilibrium cannot exist.*

## 4.1 Low risk aversion and the speculative volatility channel

In order to provide intuition underlying result **(a)**, consider first the limiting case when agents are risk neutral ($\gamma = 0$). Then the felicity function $F(C, \nu)$ in (5) is linear in $C$, and agents choose to play a one-shot lottery with all their wealth. The more optimistic agent wins in states with a high realization of the next-period aggregate endowment, while the other agent wins in states with a low realization. The cutoff is determined so that both agents are willing to participate (the agent with more wealth faces a higher probability of winning). After this one-shot lottery, the losing agent immediately becomes extinct, consuming zero at all subsequent dates.[11]

When the agents are close to risk neutral ($\gamma \searrow 0$), neither of the agents becomes extinct in finite time, but the same force, reflecting the *speculative volatility channel*, dominates the long-run dynamics. Low risk aversion incentivizes risk taking, and agents choose 'speculative' portfolios with volatile returns that reflect the differences in their assessment of probabilities of future states. While the optimal Markowitz (1952)–Merton (1971) portfolio choice is determined by the tradeoff between the expected *level* return and the underlying volatility, survival chances depend on the expected *logarithmic* growth rate of wealth, and thus on the expected *logarithmic* return on the agent's portfolio. Due to Jensen's inequality, volatile portfolios are detrimental to survival.

[11]This result is closely related to the exact role of the IES parameter, which captures the elasticity of substitution between current consumption and the expected risk-adjusted continuation value, immediately apparent from the discrete-time specification in Epstein and Zin (1989). When IES is finite ($\rho > 0$), then the only way how to optimally provide zero consumption in the next instant is to also provide zero continuation value in the same state, which also implies zero consumption at all subsequent dates and states (up to a set of paths of measure zero). A very similar mechanism underlies the results in Backus, Routledge, and Zin (2008).

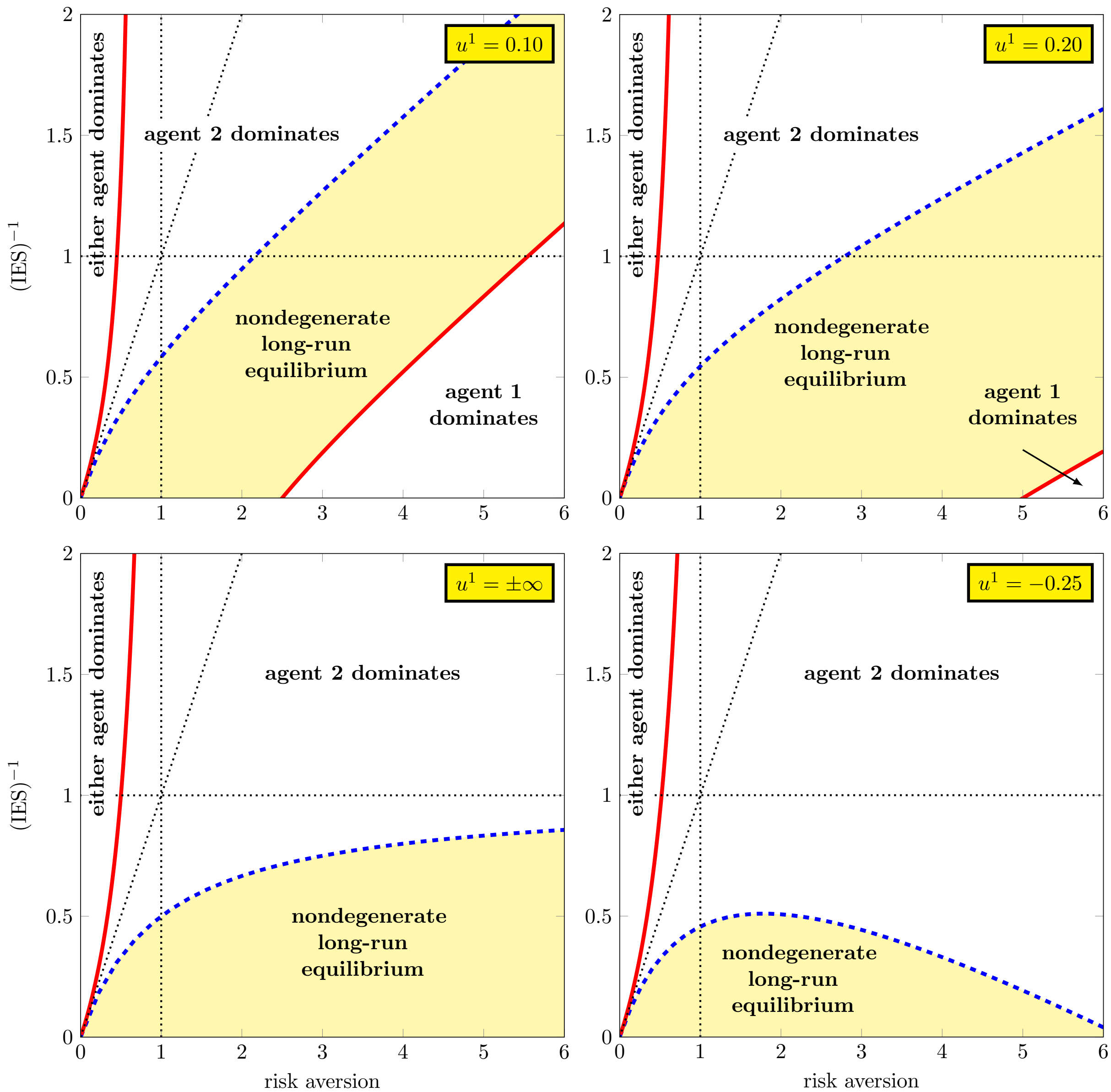


Figure 2: Survival regions for different belief distortions of agent 1 (see legend of each plot). Agent 1 perceives the drift of the aggregate endowment process to be $\mu_y + u^1\sigma_y$. Agent 2 always has correct beliefs, $u^2 = 0$, and the volatility of aggregate endowment is $\sigma_y = 0.02$.

In equilibrium, once a sequence of unsuccessful bets reduces the wealth of one agent substantially, asset prices have to adjust to induce the large agent to hold approximately the market portfolio, which prevents her from taking risky asset positions with volatile returns. At these prices, the negligible agent chooses an investment portfolio that overweighs positions in assets that are, according to her own beliefs, cheap and earn high expected *level* returns relative to their risk. When risk aversion is low, this 'speculative' position in the negligible agent's portfolio is large, the portfolio return volatile, and the expected *logarithmic* return on such a portfolio low. This leads to her extinction on a set of paths that has a strictly positive measure. Since events when either of the agents becomes sufficiently small recur with probability one, ultimately one of the agents becomes extinct.

The equilibrium price dynamics thus generate a diverging force through the speculative volatility channel. The wealth dynamics in Proposition 3.4 are dominated by the term $-\frac{1}{2}\left[\left(u^1-u^2\right)/\gamma\right]^2$ in the volatility penalty. This term is always negative and therefore drives the wealth distribution toward the boundary, irrespective of the identity of the agent. More precisely, holding other parameters fixed, there is always a level of risk aversion sufficiently low such that, depending on the sequence of shock realizations, one of the agents vanishes in the long run and each of the agents faces a strictly positive probability of extinction.

## 4.2 High risk aversion and the risk premium channel

In the other limit, when agents become highly risk averse ($\gamma \nearrow \infty$, result **(b)**), they put a high value on insuring states with low aggregate shock realizations. Since the relatively more pessimistic agent places a higher probability on these states, she is insured in equilibrium by the relatively more optimistic agent who holds a larger share of her wealth in the risky asset. The price of this insurance is the foregone risk premium in the risky asset.

Expression (11) shows that as risk aversion $\gamma$ increases, the risk premium grows linearly, and the pessimistic agent responds by insuring less the adverse states. This is reflected in the vanishing difference in the portfolio positions. However, the total cost of this insurance, given by the product of the risk premium and the difference in the portfolios, converges to a nonzero constant. Since the portfolio positions of the two agents converge to each other as risk aversion increases, their volatilities converge as well, and the volatility penalty vanishes. For high risk aversion levels, the *risk premium channel* dominates, and the more optimistic agent earns a strictly higher expected logarithmic return on her portfolio.

Equilibrium asset price dynamics play a crucial role in how this channel helps preserve long-run heterogeneity. Pick an economy where a more optimistic agent 1 survives in the long run (in the top two panels in Figure 2, those are economies to the right of the dashed blue line, which also satisfy condition **(i)** from Proposition 3.2). As the more optimistic agent accumulates a larger wealth share, the risk premium declines and the risk premium channel weakens. The general equilibrium price dynamics act as a balancing force, slowing down the rate of wealth accumulation of the optimistic agent. For a nontrivial set of moderately high values of the risk aversion parameter, this mechanism preserves a nondegenerate wealth distribution in the long run.

## 4.3 IES and the saving channel

Result **(c)** highlights the role of the *saving channel* for wealth accumulation, operating through differences in consumption rates in Proposition 3.4. Under a high IES, this channel favors the survival of the negligible agent when it induces her to choose a lower consumption rate in response to a high subjective expected portfolio return relative to the large agent.

Whenever an agent's wealth share approaches one, the market clearing mechanism forces prices to adjust so that she holds approximately the market portfolio. At these prices, the negligible agent can choose a 'speculative' portfolio invested in assets she believes are underpriced that earns a high *subjective* expected return. This also means that risk aversion cannot be too high for this mechanism to be sufficiently strong. A high risk aversion discourages speculation, and the incentives of the negligible agent to choose a sufficiently 'leveraged' portfolio with a high subjective expected return diminish.

Under a high IES, equilibrium asset price dynamics via the saving channel therefore again generate a stabilizing force that contributes to long-run survival of both agents and existence of a nondegenerate long-run wealth distribution. Proposition 3.4 shows that at the boundary, the difference in consumption rates when one agent is negligible is given by the difference in the subjective expected returns on their portfolios, scaled by the term $(1-\rho)/\rho = \text{IES} - 1$. Agents with incorrect beliefs can therefore 'outsave' their extinction when they believe that the expected return on their portfolio is high, even if it is low under the true probability distribution.

Result **(d)** is a direct counterpart to **(c)**. When preferences of the agents become inelastic ($\rho \nearrow \infty$), formulas in Proposition 3.4 imply that the survival conditions **(i)** and **(ii)** from Proposition 3.2 cannot hold simultaneously. Agents with inelastic preferences *decrease* their saving rate in response to a higher subjective expected return on their portfolio, and the saving channel operates in the opposite direction, as an extinction force for the small agent.

## 4.4 Asymmetry between optimistic and pessimistic distortions

Survival chances of agents endowed with separable preferences depend solely on the accuracy of their beliefs (Sandroni (2000), Blume and Easley (2006), Yan (2008)). Under recursive preferences, this is no longer true, and long-run wealth accumulation of optimists and pessimists differs. This is the consequence of asymmetric effects of optimistic and pessimistic beliefs on portfolio choice and saving decisions, which recursive preferences allow to separate. In this section, I study in detail the contribution of the individual survival channels to these outcomes.

### 4.4.1 Isolating the survival channels

Alternative parameter combinations allow us to isolate the three survival channels. When IES $\rho^{-1} = 1$ , agents' saving rates are both equal to $\beta$, and the saving channel is inoperational. The top two panels in Figure 2 show that the risk premium channel alone can lead to long-run survival of an incorrectly optimistic agent 1 when risk aversion is sufficiently high. However, in line with previous discussion, it cannot generate survival of an incorrectly pessimistic agent in the presence of an agent with correct beliefs (bottom right panel).

When $\sigma_y = 0$, the rational risk premium is zero and the risk premium channel is shut down. I explain in Section 4.4.3 that survival outcomes are in this case described by the bottom left panel of Figure 2. When, in addition, risk aversion is high, the speculative volatility channel is mute as well, and the isolated saving channel generates long-run survival of the incorrect agent 1 if and only if IES $> 1$.

When IES $= 1$ in addition to $\sigma_y = 0$, only the speculative volatility channel is present and an agent with incorrect beliefs can never survive with probability one in the presence of a correct agent. When the correct agent is large, risk premia are zero, so the choice of a volatile speculative portfolio by the negligible agent can only lead to a loss in terms of the expected logarithmic return due to the volatility penalty. Since the saving rates are constant as well, the speculative volatility channel is the sole force contributing to the extinction of the incorrect agent.

### 4.4.2 Optimistic belief distortion

We can now more systematically explore Figure 2. The first panel starts with a moderately optimistic agent 1. The correct agent 2 dominates in the long run in the neighborhood of the dotted diagonal, extending the results for the CRRA case continuously in the parameter space. The graph also confirms all four asymptotic results from Corollary 4.1.

At the same time, there is a nontrivial intermediate region (depicted as shaded in the graph) where both agents coexist in the long run. In this whole region, risk aversion is larger than the inverse of IES, which is a standard parametric choice in the asset pricing literature. The two boundaries in the top left panel which delimit this region are asymptotically parallel as $\gamma \nearrow \infty$ with slope $2\sigma_y / \left(u^1 + u^2 + 2\sigma_y\right)$.

As optimism of agent 1 increases (second panel in Figure 2), the lines delimiting the shaded region rotate clockwise. The area in which agent 2 dominates expands, reflecting an increase in inaccuracy of agent's 1 beliefs, but the region in which both agents coexist never vanishes.

In fact, as $u^1 \nearrow \infty$ and agent 1 becomes infinitely optimistically biased, we obtain the third panel in Figure 2.[12] The optimistic agent 1 never dominates the economy but there is a large set of parameter combinations for which both agents coexist in the long run. The dashed line delineating this set converges to IES = 1 as risk aversion increases. The shaded region includes plausible parameterizations used in asset pricing; for example, much of the long-run risk literature initiated by Bansal and Yaron (2004) advocates IES significantly above one and risk aversion above five.

### 4.4.3 Economy with constant aggregate endowment

In economies in the bottom left panel of Figure 2, incentives to speculate, driven by the magnitude of the difference in belief distortions, are arbitrarily large relative to aggregate risk in the economy. Given $u^2 = 0$, we have seen in Section 3.3 that survival results do not depend on $u^1$ and $\sigma_y$ independently but only on the ratio $u^1/\sigma_y$. Survival results for the case $u^1/\sigma_y \to \infty$ thus equivalently describe economies with $u^1 \to \infty$ or $\sigma_y \to 0$.

To illuminate this mechanism, consider the limiting case of an economy without aggregate risk, $\sigma_y = 0$, with $u^1$ being an arbitrary nonzero belief distortion and $u^2 = 0$. Since survival results do not depend on $\mu_y$, we can take $\mu_y = 0$ and hence consider an economy with constant aggregate endowment, $Y_t = \overline{Y}$. Agents in this economy trade for purely speculative motives, see, e.g, Brunnermeier, Simsek, and Xiong (2014). Importantly, this experiment also shows that the survival results in this paper do not hinge upon the economy being unbounded (see Section 6 for a

[12] Increasing belief biases may, depending on the parameter configuration, violate Assumption A.1 and an equilibrium may cease to exist if agents are not sufficiently patient. However, as we will see, the same shift toward the third panel can be achieved by holding the belief bias fixed while decreasing aggregate volatility, $\sigma_y \searrow 0$.

more detailed discussion) and are driven solely by the characteristics of Epstein–Zin preferences.[13]

The long-run survival results for this economy are perfectly equivalent to the bottom left panel in Figure 2. As agent 1 becomes negligible, agent 2 has to hold the market portfolio. Since this portfolio corresponds to a claim on the deterministic consumption stream, her consumption becomes deterministic and risk premia converge to zero (see also the pricing results in Section 5). From the perspective of agent 1, the claim on $W$ now offers a high perceived return and, with IES $> 1$, this translates into a higher saving rate of the negligible agent. When IES is sufficiently high, the high saving motive is always strong enough to let the negligible agent outsave her extinction and survive in the long run. Section 4.5 also analyzes this economy under symmetric belief distortions.

#### 4.4.4 Pessimistic belief distortion

The third panel in Figure 2 also represents the case when $u^1/\sigma_y \to -\infty$, i.e., the case of an infinitely pessimistic agent 1. Recall that the limit $\left|u^1/\sigma_y\right| \to \infty$ corresponds to a situation where the role of aggregate risk vanishes relative to the speculative motives generated by belief heterogeneity. In this limit, the agents are speculating on the realizations of the Brownian shock $W$ without distinguishing 'good' and 'bad' aggregate states. Because this shock is symmetric, it does not matter whether agent 1 is 'optimistic' and speculates on right-tail realizations of the Brownian shock or 'pessimistic' and speculates on left-tail realizations. This logic is most clearly visible in the case with deterministic aggregate endowment, $\sigma_y = 0$, where the survival results are the same for an arbitrary value of $u^1 \neq 0$.

What happens when the magnitude of pessimism decreases and $u^1$ starts moving from $-\infty$ closer to zero? The change in the survival regions is represented by a move from the third to the fourth panel of Figure 2. The region in which the pessimistic agent 1 survives actually *shrinks*.

Since the pessimistic agent invests a smaller share of her wealth in the risky asset, she cannot benefit from the risky asset's higher expected return through the risk premium channel. At the same time, a long position in the risky asset would also imply that her *subjective* expected return is even lower and she will not improve her survival chances by choosing a higher saving rate under IES $> 1$. However, a sufficiently strong speculative motive induces the pessimistic agent to short the risky asset, and makes her in fact *optimistic* about the return on such a portfolio. As we will see in Section 5, the term in brackets in the consumption rate difference (12), which dominates the saving decisions when $\rho \searrow 0$, is equal to

$$\left(u^1 - u^2\right) \sigma_y \left(1 + \pi^1\left(0\right)\right) \tag{13}$$

where $\pi^1\left(0\right)$ is equal to agent 1's risky portfolio share. If agent 1 is relatively more pessimistic, then $u^1 - u^2 < 0$, and thus $\pi^1\left(0\right) < -1$ is needed for the saving motive of agent 1 to dominate that

[13] The case without aggregate risk requires a clarification regarding the contract space in the decentralized economy. A natural decentralization in the economy with $\sigma_y > 0$ involves a claim to aggregate endowment with unit supply and an infinitesimal risk-free claim in zero net supply. In order to allow agents to trade on their heterogeneous beliefs regarding the probability distribution of the Brownian motion $W$ when $\sigma_y = 0$, a suitable decentralization involves a claim on $W$ in zero net supply and a risk-free claim with supply $\overline{Y}$. This decentralization is explained in more detail in Section OA.3.3 of the Online Appendix.

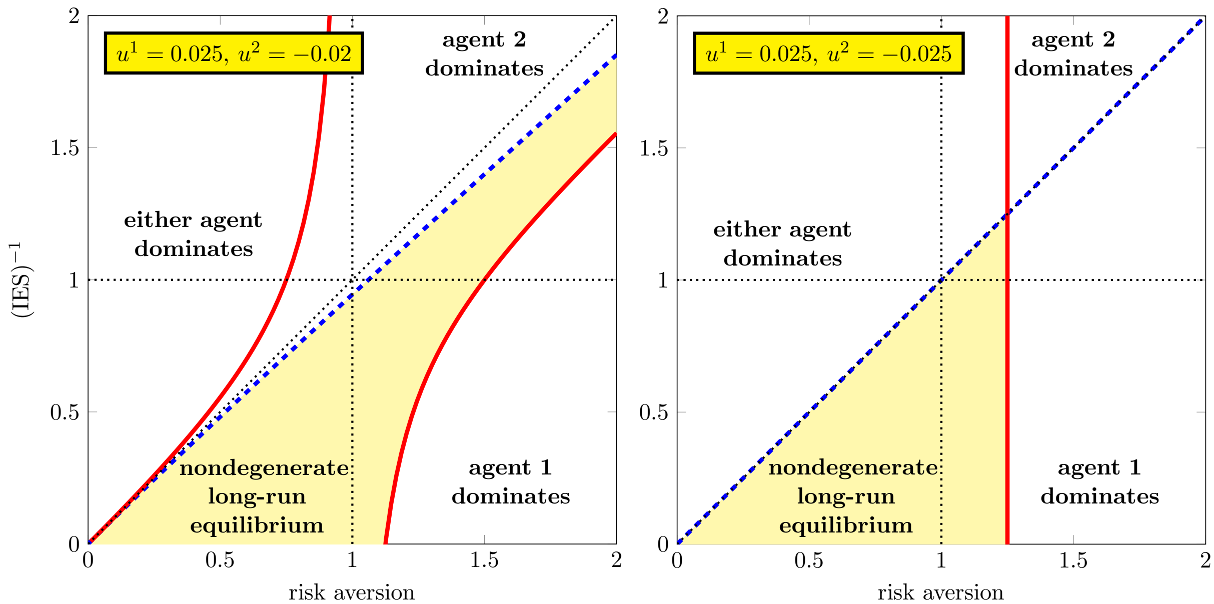


Figure 3: Survival regions for an optimistic agent 1 and a pessimistic agent 2 (see legend of each plot). Agents perceive the drift of the aggregate endowment process to be $\mu_y + u^n\sigma_y$. Volatility of aggregate endowment is $\sigma_y = 0.02$.

of the large agent 2 as $\rho \searrow 0$. While the short position in the risky asset earns a low objective expected return, a high IES can generate a sufficiently strong offsetting saving motive that will allow the pessimistic agent to outsave her extinction.

The region in the fourth graph in which the two agents coexist does not include high levels of risk aversion and shrinks for smaller belief distortions. A high level of risk aversion or a lower incentive to speculate caused by a smaller belief distortion prevent the small agent from choosing a sufficiently large short position in the risky asset which is, as shown in formula (13), necessary to generate the high subjective expected return needed for the saving mechanism to operate in favor of the pessimistic agent 1.

The above discussion also explains why the described mechanism cannot lead to the long-run dominance of the pessimistic agent. As the wealth share of the pessimistic agent approaches one, she can no longer hold a short position in the risky asset, and the effect of the saving channel generated through the high subjective expected return disappears.

### 4.5 Symmetric belief distortions

Figure 2 analyzes economies where agent 2 has correct beliefs. Another interesting case arises when the two agents have equal magnitudes of belief biases with opposite signs. Figure 3 captures the case of an optimistic agent 1 ($u^1 = 0.025$) and a pessimistic agent 2. In the left panel, beliefs of agent 2 are somewhat less biased ($u^2 = -0.02$), while in the right panel, the magnitude of belief biases is equal for both agents.

As in the previous analysis, all four combinations of survival outcomes occur in relevant parts of the state space. In the right panel, the parameter space is exactly separated along the CRRA preference parameterizations (dotted blue line), and along the solid red vertical line that lies at the level of risk aversion equal to $u^1/\sigma_y$. Therefore, as the magnitude of the belief distortion $u^1 = -u^2$ increases, or as aggregate uncertainty vanishes and we converge to the setup from Section 4.4.3, the vertical line shifts to the right and the region with a nondegenerate long-run equilibrium expands. Section OA.4 in the Online Appendix provides more detail.

## 4.6 Separable preferences

The framework introduced in this paper includes separable CRRA preferences as a special case. Yan (2008) and Kogan, Ross, Wang, and Westerfield (2017) show that under identical CRRA preferences, the agent whose beliefs are less distorted dominates in the long run under measure $P$. The conditions in Proposition 3.2 imply the following Corollary that confirms these results:

**Corollary 4.2** *Under separable CRRA preferences ($\gamma = \rho$), agent $n$ dominates in the long run under measure $P$ if and only if $|u^n| < |u^{\sim n}|$. Agent $n$ survives under $P$ if and only if the inequality is non-strict. Further, agent $n$ always survives under measure $Q^n$, and dominates in the long run under $Q^n$ if and only if $u^n \neq u^{\sim n}$.*

In the separable case, the dynamics of the Pareto share $\theta$ do not depend on the characteristics of the endowment process. The relative Pareto weight $\vartheta$ in (10) is a Brownian motion with a constant drift $\frac{1}{2}\left[\left(u^2\right)^2 - \left(u^1\right)^2\right]$ and therefore diverges to $+\infty$ or $-\infty$ ($P$-a.s.) depending on relative magnitudes of the belief distortions, implying that the agent with a lower magnitude of the belief distortion $|u^n|$ dominates.

This result also provides a consistency check of Proposition 3.2 with the analysis of survival in Blume and Easley (2006) under separable preferences and more general stochastic environments. That paper uses relative entropy, or the Kullback–Leibler divergence, of the subjective belief $Q^n$ relative to the data-generating measure $P$ as a statistic that summarizes the contribution of belief distortions to long-run survival. In the continuous-time Brownian information setup, the increment of relative entropy of agent's $n$ subjective belief is $\frac{1}{2}|u^n|^2\,dt$. In the separable case, it is indeed the agent with lower relative entropy who dominates.[14] When preferences are not separable, the discount rates $\nu^n$ in (10) are endogenously determined and relative entropy is not a summary statistic for the determination of survival. Section OA.4.5 in the Online Appendix elaborates.

## 4.7 Long-run consumption distribution

Propositions 3.2 and 3.4 derive parametric restrictions on the survival regions. However, even if a nondegenerate long-run equilibrium exists, the question remains whether this equilibrium delivers quantitatively interesting endogenous dynamics under which each of the agents can gain a significant wealth share. We start with the following observation.

[14] A special case emerges when $u^2 = -u^1 \neq 0$, corresponding to equilibria along the dotted blue line in the right panel of Figure 3. In this situation with CRRA preferences and symmetric distortions, none of the agents becomes extinct but nonetheless a nondegenerate long-run distribution for the Pareto share does not exist (these results follow from the observation that in this case, the relative Pareto share $\vartheta$ in (10) is a Brownian motion without drift). More detail is provided in the proof to Corollary 4.2.

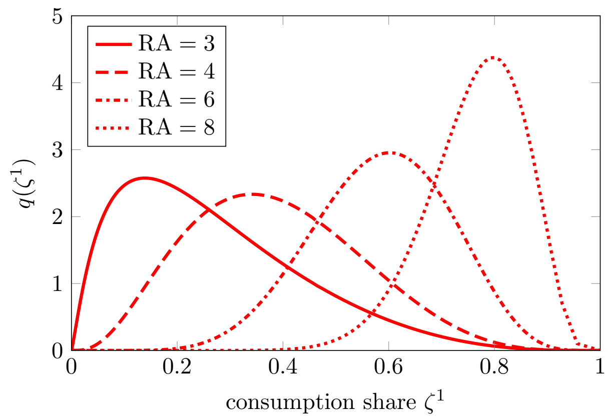


Figure 4: Stationary distributions for the consumption share $\zeta^1(\theta)$ of the agent with optimistically distorted beliefs. All models are parameterized by $u^1 = 0.25$, $u^2 = 0$, IES $= 1.5$, $\beta = 0.05$, $\mu_y = 0.02$, $\sigma_y = 0.02$, and differ in levels of risk aversion, shown in the legend.

**Proposition 4.3** *When agent $n$ survives in the long run, she attains an arbitrarily large wealth share $A^n/A \in (0,1)$ and consumption share $\zeta^n \in (0,1)$ with probability one at some future date $t$.*

This result is a consequence of sufficient mixing in the Pareto share process $\theta$. However, assessing whether the agent also *typically* holds a large wealth share requires a full numerical solution of the model in the interior of the state space. Figure 4 shows the densities $q\left(\zeta^1\right)$ for the long-run distribution of the consumption share $\zeta^1$ in example economies where both agents survive, for the case of an optimistic agent 1 and correct agent 2 and alternative levels of risk aversion. Proposition 4.3 already revealed that these densities have a full support on $(0,1)$.

The graphs in Figure 2 showed that increasing risk aversion improves the survival chances of the optimistic agent 1. Figure 4 provides a complementary perspective. As we increase risk aversion, the distribution of the consumption share shifts in favor of the optimistic agent, due to the stronger risk premium channel.

Several observations emerge. First, when both agents survive in the long run, the more incorrect agent can plausibly own and consume a substantial share of aggregate endowment. Second, the shape of the stationary densities for the consumption share indicates that long-run equilibria permit substantial variation over time in these consumption shares. Finally, the same survival channels that generate different types of long-run survival outcomes also act in favor of individual agents within the interior of the state space. A more detailed analysis of these mechanisms as well as evolution of the consumption distribution over time can be found in Online Appendix OA.5.

## 5 Asset prices and price impact

Kogan, Ross, Wang, and Westerfield (2006, 2017) distinguish between survival of agents with incorrect beliefs and their impact on equilibrium prices. This leads to the following two distinct questions. First, do agents with incorrect beliefs have an impact on asset prices in the long run? Second, if an agent currently holds a negligible wealth share, does she have any impact on *current* asset prices and returns, even if she potentially survives in the long run?

The results in this section reveal that as the Pareto share of one of the agents becomes negligible, current asset prices and infinitesimal asset returns converge to those which prevail in a homogeneous economy populated by the agent with the large Pareto share, regardless whether the negligible agent ultimately survives or vanishes. This directly implies that an agent who becomes extinct in the long run also has no long-run *price impact*. On the other hand, an agent who survives will have an impact on asset prices in the future when her wealth share recovers, even if her current wealth share and price impact may be negligible.

The ability to pin down asset returns when the wealth of one agent is negligible plays a crucial role in establishing the analytical results in Proposition 3.4 because it allows me to determine the wealth dynamics of the two agents in the proximity of the boundary by solving two straightforward portfolio choice problems.

The following Proposition summarizes the limiting pricing implications as the wealth share of one of the agents becomes arbitrarily small. Without loss of generality, it is sufficient to focus on the case $\theta \searrow 0$.

**Proposition 5.1** *As $\theta \searrow 0$, the infinitesimal risk-free rate $r(\theta)$, the aggregate wealth-consumption ratio $\xi(\theta)$, and the coefficients of the aggregate wealth process $d\log A_t = \mu_A(\theta_t)\,dt + \sigma_A(\theta_t)\,dt$ converge to their homogeneous economy counterparts:*

$$\lim_{\theta\searrow 0} r(\theta) = r(0) = \beta + \rho\left(\mu_y + u^2\sigma_y + \frac{1}{2}(1-\gamma)\sigma_y^2\right) - \frac{1}{2}\gamma\sigma_y^2,$$

$$\lim_{\theta\searrow 0} \xi(\theta) = \xi(0) = \left[\beta - (1-\rho)\left(\mu_y + u^2\sigma_y + \frac{1}{2}(1-\gamma)\sigma_y^2\right)\right]^{-1},$$

$$\lim_{\theta\searrow 0} \mu_A(\theta) = \mu_y, \quad \text{and} \quad \lim_{\theta\searrow 0} \sigma_A(\theta) = \sigma_y.$$

*Consequently, the infinitesimal logarithmic return on the claim on aggregate wealth,*

$$d\log R_t \doteq \left[[\xi(\theta_t)]^{-1} + \mu_A(\theta_t)\right] dt + \sigma_A(\theta_t)\,dW_t, \tag{14}$$

*has coefficients that converge as well.*

The proof is provided in Appendix B and is based on the characterization of the dynamics of the equilibrium stochastic discount factor as $\theta \searrow 0$. Marginal utility under recursive preferences is forward-looking and depends on agent's continuation value (see the stochastic discount factor specification (22)), so that future equilibrium consumption dynamics affect the current local evolution of the stochastic discount factor. A crucial step involves showing that the dynamics of the relative Pareto share $\vartheta(\theta)$ in (10) have bounded drift and volatility coefficients. This implies that the rate of wealth accumulation of the negligible agent is sufficiently slow so that even in the case she survives, her potential future impact on the economy is sufficiently distant to be inconsequential for the current evolution of the stochastic discount factor and asset prices. In addition, the agent with negligible wealth has no current price impact not only on the two assets that dynamically complete the market but also on every finite-maturity bond and consumption strip.

**Corollary 5.2** *For every fixed maturity $t$, the prices of a zero-coupon bond and a claim to a payout from the aggregate endowment stream (a consumption strip) converge to their homogeneous economy counterparts as $\theta \searrow 0$.*

### 5.1 Decision problem of an agent with negligible wealth

Proposition 5.1 establishes that the actual *general equilibrium* price dynamics and choices of the large agent in the proximity of the boundary are locally the same as those in an economy populated only by the large agent 2. To conclude the argument, we need to infer the wealth dynamics for agent 1 that has negligible wealth. As in the case of the large agent, the impact of future equilibrium consumption dynamics on the current decisions of the negligible agent becomes immaterial as $\theta \searrow 0$, despite the nonseparability of preferences.

**Proposition 5.3** *The consumption-wealth ratio of agent 1 converges to*

$$\lim_{\theta \searrow 0} \left[\xi^1(\theta)\right]^{-1} = [\xi(0)]^{-1} - \frac{1}{2}\frac{1-\rho}{\rho}\left[2\left(u^1 - u^2\right)\sigma_y + \frac{\left(u^1-u^2\right)^2}{\gamma}\right] \tag{15}$$

*and the wealth share invested into the claim on aggregate consumption to*

$$\lim_{\theta \searrow 0} \pi^1(\theta) = 1 + \frac{u^1 - u^2}{\gamma \sigma_y}. \tag{16}$$

Proposition 5.3 derives the consumption-saving decision (15) and portfolio allocation decision (16) relative to the same decisions of the large agent 2. Recall that agent 2's choices agree in the limit as $\theta \searrow 0$ with aggregate ones—equilibrium prices have to adjust so that her consumption-wealth ratio is equal to the aggregate ratio $[\xi(0)]^{-1}$ and she holds the market portfolio, $\pi^2(0) = 1$.

At these equilibrium prices, agent 1's consumption and portfolio choices deviate from the aggregate dynamics according to formulas (15) and (16). When these deviations lead to a high saving rate or a high logarithmic return on agent 1's portfolio, they can prevent her extinction. As the formulas indicate, when $u^1 = u^2$ the agents are identical and their decisions and wealth dynamics coincide. A relatively more optimistic agent ($u^1 > u^2$) chooses a larger share of her wealth to be invested in the risky asset, and chooses to save more (less) when IES $> 1$ ($< 1$).

These portfolio and consumption-saving decision of agent 1 as $\theta \searrow 0$ coincide with a 'partial equilibrium' solution where agent 1 locally behaves as if she lived forever as an infinitesimal agent in a homogeneous economy populated only by the large agent 2. The logic of the proof relies on showing that by pushing the current $\theta$ arbitrarily close to zero, one can extend the time before the presence of the agent 1 becomes noticeable from aggregate perspective (measured, e.g., by sufficiently large deviations in prices or return distributions from their homogeneous economy counterparts) arbitrarily far into the future.

This implies that the survival question, whose answer only depends on the behavior at the boundaries, can be resolved by studying homogeneous economies with an infinitesimal price-taking agent. Even if the negligible agent survives with probability one and has an impact on equilibrium prices in the long run, these effects do not influence *current* prices, returns, and wealth dynamics.

## 6 Methodology and literature overview

The modern approach in the market survival literature originates from the work of De Long, Shleifer, Summers, and Waldmann (1991), who study wealth accumulation in a partial equilibrium setup with exogenously specified returns and find that irrational noise traders can outgrow their rational counterparts and dominate the market. Similarly, Blume and Easley (1992) look at the survival problem from the vantage point of exogenously specified saving rules, albeit in a general equilibrium setting.[15]

Subsequent research has shown that taking into account general equilibrium effects and intertemporal optimization of agents endowed with separable preferences eliminates much of the support for survival of agents with incorrect beliefs that models with ad hoc price dynamics produce. Sandroni (2000) and Blume and Easley (2006) base their survival results on the evolution of relative entropy as a measure of disparity between subjective beliefs and the true probability distribution. In their work, aggregate endowment is bounded from above and away from zero. As a result, local properties of the utility function are immaterial for survival. Controlling for pure time preference, the long-run fate of economic agents depends solely on belief characteristics, and only agents whose beliefs are in the relative entropy sense asymptotically closest to the truth survive.

With unbounded aggregate endowment, the local properties of the utility function become an additional survival factor. Even if preferences are identical across agents, the local curvature of the utility function at low and high levels of consumption can be sufficiently different to outweigh the divergence in beliefs and lead to the survival of agents with relatively more incorrect beliefs. Kogan, Ross, Wang, and Westerfield (2017) show elegantly that a sufficient condition to prevent this outcome is the boundedness of the relative risk aversion function, i.e., a condition on the preferences being uniformly 'close' to the homothetic CRRA case.[16] In contrast, in my paper preferences are homothetic, which assures that the survival results are not driven by exogenous differences in the local properties of the utility functions. Section 4.4.3 also documents that unbounded aggregate endowment is not essential for the survival mechanisms analyzed in this economy.

Since survival results under separable preferences depend on pairwise comparisons of relative entropy across agents, the analysis directly applies to multiple classes of agents (Massari (2016) considers a continuum of agent types) and can involve richer belief dynamics (see Section OA.4.5 in the Online Appendix for more discussion on this approach). Incorporating such features in my framework with recursive preferences increases the complexity of the problem in nontrivial ways.

[15] A thorough review of the quickly growing literature on models with heterogeneous beliefs is beyond the scope of this paper. Instead, I focus on the intersection of this literature with the analysis of nonseparable preferences. Bhamra and Uppal (2014) provide a more general survey that also focuses on asset pricing implications of belief and preference heterogeneity. I also omit the discussion of evolutionary literature which predominantly studies the interaction of agents with exogenously specified portfolio rules and price dynamics. See Hommes (2006) for a survey of the evolutionary literature, and Evstigneev, Hens, and Schenk-Hoppé (2006) for an analysis of portfolio rule selection.

[16] The survival results under separable utility thus also hold for 'exotic' endowment processes like the rare disaster framework in Chen, Joslin, and Tran (2012). The survival literature also studies other forms of heterogeneity. Yan (2008) and Muraviev (2013) construct 'survival indices' that summarize the contribution of belief distortions and preferences to survival chances. Market incompleteness or asymmetric information are other ways how to counteract the extinction of agents with incorrect beliefs, as long they are judiciously chosen to prevent agents from placing incorrect bets, see, e.g., Mailath and Sandroni (2003), Beker and Chattopadhyay (2010), Coury and Sciubba (2012), Cao (2017) or Cogley, Sargent, and Tsyrennikov (2014).

However, the results in Section 4.6 show that my approach reduces to a comparison of relative entropies when preferences are separable, in line with the above papers.

Closely related to consumption dynamics is the evolution of equilibrium asset prices and agents' portfolio choices in the decentralized economy. Kogan, Ross, Wang, and Westerfield (2006) explain in detail the distinction between the impact agents with incorrect beliefs have on consumption dynamics and on asset prices, while Cvitanić and Malamud (2011) also focus on impact on portfolio choices. Both papers analyze finite horizon economies with only terminal consumption, and long-run outcomes are studied from the perspective of a sequence of such economies as the terminal date diverges to infinity. This approach is substantially different from studying an infinite horizon economy with intertemporal consumption decisions, and specific results and insights may not be transferrable across the two frameworks. I devote Section OA.6 of the Online Appendix to a more detailed comparison of Kogan, Ross, Wang, and Westerfield (2006) with the results in my paper.

Survival analysis under separable preferences corresponds to studying a sequence of time- and state-indexed static problems that are only interlinked through the initial marginal utility of wealth. Nonseparability of preferences breaks this straightforward link, and I therefore develop a different method that is more suitable for this environment. I utilize the planner's problem derived in Dumas, Uppal, and Wang (2000) and extend it to include heterogeneity in beliefs. While the analysis under separable preferences reflects the purely *intratemporal* tradeoff in the allocation of consumption vis-à-vis changes in the local curvature of the period utility function, the nonseparable nature of recursive preferences introduces an additional *intertemporal* component captured in the dynamics of the Pareto weights.

The approach based on the characterization of endogenously determined Pareto weight dynamics is closely linked to the literature on endogenous discounting, initiated by Koopmans (1960) and Uzawa (1968), and to models of heterogeneous agent economies under recursive preferences, studied by Lucas and Stokey (1984) and Epstein (1987) under certainty and by Kan (1995) under uncertainty. In particular, Lucas and Stokey (1984) impose an exogenous restriction on the preference specification, which they call increasing marginal impatience, that is sufficient for the existence of a stable interior steady state in their framework. This condition requires that preferences are nonhomothetic, and agents must discount future more as they become richer. The key difference to the conditions based on *relative patience* derived in my paper lies in the determination of the two quantities. While Lucas and Stokey require that the time preference exogenously encoded in the utility specification changes with the level of consumption, in my paper preferences are homothetic and the variation in relative patience arises endogenously as a response to the equilibrium price dynamics driven by belief differences.

Anderson (2005) studies Pareto optimal allocations under heterogeneous recursive preferences in a discrete-time setup using similar methods but he does not investigate survival under belief heterogeneity. Beker and Espino (2011) consider portfolio choice under separable preferences. Mazoy (2005) discusses long-run consumption dynamics when agents differ in their IES. Colacito, Croce, and Liu (2017) prove the existence of nondegenerate long-run equilibria in a two-good economy when agents are endowed with risk-sensitive preferences and differ in the preferences over the two goods. Branger, Dumitrescu, Ivanova, and Schlag (2011) analyze survival in long-run risk models with heterogeneous recursive preferences. Finally, Dindo (2015) formulates a discrete-time problem analogous to the one studied in this paper, and is able to derive analytical results for particular configurations of preference parameters using different techniques, confirming that the survival results are not specific to the continuous-time Brownian information framework.

Subjective beliefs may also emerge endogenously from preferences featuring aversion to model uncertainty. In particular, Guerdjikova and Sciubba (2015) study a heterogeneous-preference economy populated by expected-utility and smooth ambiguity averse agents. The ex-post behavior of a smooth ambiguity averse agent corresponds to one with an endogenously determined subjective belief and discount rate. Their paper shows that smooth ambiguity averse agents can dominate in the economy if their preferences satisfy a sufficiently strong decreasing absolute ambiguity aversion condition. Under this condition, the preferences exhibit non-homotheticity that makes agents overweigh disproportionately more pessimistic scenarios, hence endogenously increasing the precautionary motive and making the agent effectively more patient. Contrary to this, preferences in my paper are homothetic and identical across agents. Moreover, Guerdjikova and Sciubba (2015) do not identify cases where both types of agents coexist in the long run, nor do they analyze economic decisions in the decentralized economy. Connections to results in my paper, including the continuous-time limit of smooth ambiguity averse preferences in the sense of Skiadas (2013), are more closely explored in Section OA.7.3 of the Online Appendix. In a similar vein, Bhandari (2015) studies a problem where heterogeneous beliefs arise in a setup with robust preferences of Hansen and Sargent (2001). In this case, ex-post behavior is characterized only by belief distortions.

A large literature documents heterogeneity in beliefs of economic agents (Mankiw, Reis, and Wolfers (2003), Malmendier and Nagel (2016)), and its consequences for macroeconomic and asset price dynamics (Buraschi and Jiltsov (2006), Bachmann, Elstner, and Sims (2013), Giacoletti, Laursen, and Singleton (2015), or Barillas and Nimark (2017); see also Xiong (2013) for a survey in the asset pricing context). While I remain agnostic about the source of this belief heterogeneity and focus on its implications on economic outcomes, possible extensions that endogenize these subjective beliefs are discussed in Section OA.7 of the Online Appendix.

# 7 Concluding remarks

This paper analyzes portfolio and consumption-saving decisions and their implications for long-run wealth dynamics in an economy populated by agents with heterogeneous beliefs. It shows that the robust survival results favoring agents with most accurate beliefs found in the existing literature are specific to the class of separable preferences. Under nonseparable recursive preferences of the Duffie–Epstein–Zin type, long-run outcomes in which heterogeneity prevails and agents with incorrect beliefs affect equilibrium dynamics arise for a broad set of plausible parameterizations when risk aversion is larger than the inverse of the intertemporal elasticity of substitution.

The analysis reveals the important distinct roles played by risk aversion with respect to intratemporal gambles that determines risk taking, and intertemporal elasticity of substitution that drives the consumption-saving decision. In particular, the paper uncovers the complex interaction between risk sharing and speculative motives of agents with heterogeneous beliefs, and their joint impact on the saving behavior. Speculative behavior by an agent with incorrect beliefs distorts her rationally optimal risk-return tradeoff. This can aid wealth accumulation if it leads the agent to choose a portfolio with a higher expected logarithmic return. Similarly, a higher *subjective* expected return implies a higher saving rate when IES is sufficiently high. Critical for obtaining the survival results, and in particular the nondegenerate long-run equilibria, are the general equilibrium price dynamics generated by shifts in the wealth distribution.

On the technical side, I provide a novel existence proof of a classical solution to the Hamilton–Jacobi–Bellman equation associated with the planner's problem and its equivalence with the planner's value function. The survival results are tightly linked to the analytical characterization of the boundary behavior of this equation and agents' portfolio and consumption choices in the associated decentralized economy. Since this type of ODE arises in a wider class of recursive utility problems, these techniques can be utilized in a broader variety of economic applications.

A growing empirical literature collects evidence on portfolio choice and saving rates (Calvet, Campbell, and Sodini (2009), Fagereng, Guiso, Malacrino, and Pistaferri (2016)) and the implications for wealth accumulation (Kopczuk (2015), Saez and Zucman (2016)) and uncovers patterns of highly heterogeneous portfolio returns that substantially impact the evolution of the wealth distribution. Kendall and Oprea (2016) complement this evidence with results from laboratory experiments that link belief biases, portfolio choice and consumption-saving decisions, supporting the hypothesis that belief heterogeneity may be an important contributing factor. My paper expounds the risk-return and consumption-saving tradeoffs in the presence of belief heterogeneity and their implications for long-run wealth dynamics. In this way, the paper contributes to the theoretical literature analyzing various factors underlying long-run wealth heterogeneity, see, e.g., Benhabib, Bisin, and Zhu (2011), Gabaix, Lasry, Lions, and Moll (2016), or the survey in Benhabib and Bisin (2016). In particular, it complements alternative models of incorrect beliefs and limited investor sophistication (Kacperczyk, Nosal, and Stevens (2015), Lusardi, Mitchell, and Michaud (2016)).

I obtain the results in a complete-market two-agent economy with an aggregate endowment process that is specified as a geometric Brownian motion. The focus of this paper is primarily theoretical and the stochastic structure of the economy is kept deliberately simple to yield a sharp analytical characterization, sacrificing quantitative fit. However, economic forces underlying the long-run wealth dynamics are not confined to this environment. For example, Baker, Hollifield, and Osambela (2016) use insights derived here to study consumption, investment and equity returns in a production economy with recursive preferences and disagreement, while Pohl, Schmedders, and Wilms (2017) analyze implications of heterogeneous beliefs about shock persistence for risk premia in a discrete-time long-run risk model, in both cases arguing that agents with incorrect beliefs can have a persistent, quantitatively important effect on the economy.

On the other hand, the complete-market framework studied here lacks other features that are plausibly relevant for quantitative work—for instance, life cycle considerations (Gârleanu and Panageas (2015), Collin-Dufresne, Johannes, and Lochstoer (2017)) or various forms of market incompleteness (Cogley, Sargent, and Tsyrennikov (2014), Cao (2017)). Even though extending the model in these directions may affect allocations in important ways, a sharp characterization of wealth dynamics in the frictionless economy remains an informative benchmark.

# Appendix

The Appendix provides proofs of the propositions from Sections 3–5. Lengthier detailed calculations are deferred to Section OA.11 of the Online Appendix. Proof of Proposition 2.3 is provided in Sections OA.9–OA.10 of the Online Appendix.

## A Parametric restrictions

A unique solution to the planner's problem exists when the following restriction on the parameters holds.

**Assumption A.1** *The parameters in the model satisfy the restrictions*

$$\beta > \max_n (1-\rho)\left(\mu_y + u^n\sigma_y + \frac{1}{2}(1-\gamma)\sigma_y^2\right), \tag{17}$$

$$\beta > \max_n (1-\rho)\left(\mu_y + u^{\sim n}\sigma_y + \frac{1}{2}(1-\gamma)\sigma_y^2\right) + \frac{1}{2}\frac{1-\rho}{\rho}\left[2\left(u^n - u^{\sim n}\right)\sigma_y + \frac{\left(u^n - u^{\sim n}\right)^2}{\gamma}\right] \tag{18}$$

*for $n \in \{1,2\}$ where $\sim n$ is the index of the agent other than $n$.*

The first restriction is sufficient for the continuation values in the homogeneous economies to be well-defined, and that the wealth-consumption ratio of the large agent is finite (Proposition 5.1). The second restriction is a sufficient condition assuring that the wealth-consumption ratio is asymptotically well-behaved when the agent becomes infinitesimally small (Proposition 5.3). Both conditions are restrictions on the time-preference parameter of the agents and can always be jointly satisfied by making the agents sufficiently impatient. For instance, when IES $= 1$, these conditions amount to $\beta > 0$. However, since the survival results do not depend on $\beta$, Assumption A.1 does not introduce substantial restrictions for the analysis of the problem. Section OA.4.1 in the Online Appendix provides more detail and graphically depicts minimal values of $\beta$ that are sufficient for existence and uniqueness across the parameter space.

## B Characterization of the boundary behavior

This section contains proofs of propositions that characterize the boundary behavior of the economy as $\theta \to \{0,1\}$. I start with Proposition 3.2, then move ahead to prove propositions from Section 5 and finally return to prove the remaining statements.

**Proof of Proposition 3.2.** Since the transformation $\vartheta(\theta)$ is smooth and strictly increasing, the process $\vartheta$ in (10) is an Itô diffusion and asymptotic properties of $\theta$ and $\vartheta$ are equivalent, with the limits $\theta \to \{0,1\}$ corresponding to $\vartheta \to \{-\infty,+\infty\}$. Given an initial condition $\vartheta_0 \in \mathbb{R}$, the process $\vartheta$ lives on $\mathbb{R}$ with unattainable boundaries $\{-\infty,+\infty\}$ (the preferences satisfy an Inada condition at zero). For any numbers $a < b$, the process $\vartheta$ has bounded and continuous drift and volatility coefficients on $(a,b)$, and the volatility coefficient is bounded away from zero. It is thus sufficient to establish the appropriate boundary behavior of $\vartheta$ in the proximity of $\{-\infty,+\infty\}$ in order to make the process positive Harris recurrent (see Meyn and Tweedie (1993)). Since the process is also $\varphi$-irreducible for the Lebesgue measure under these boundary conditions, there exists a unique stationary distribution.

Denote $\mu_\vartheta(\vartheta)$ and $\sigma_\vartheta(\vartheta)$ the drift and volatility coefficients in (10). The boundary behavior of the process $\vartheta$ is captured by the scale function $S : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$s(\vartheta) = \exp\left\{-\int_{\vartheta_0}^{\vartheta} \frac{2\mu_\vartheta(\tau)}{\sigma_\vartheta^2(\tau)} d\tau\right\} \qquad S[\vartheta_l, \vartheta_h] = \int_{\vartheta_l}^{\vartheta_h} s(\vartheta)\, d\vartheta$$

for an arbitrary choice of $\vartheta_0 \in \mathbb{R}$, and the speed measure $M : \mathbb{R}^2 \to \mathbb{R}$

$$m(\vartheta) = \frac{1}{\sigma_\vartheta^2(\vartheta)\, s(\vartheta)} \qquad M[\vartheta_l, \vartheta_h] = \int_{\vartheta_l}^{\vartheta_h} m(\vartheta)\, d\vartheta.$$

Karlin and Taylor (1981, Chapter 15) provide an extensive treatment of the boundaries. The boundaries are nonattracting if and only if

$$\lim_{\vartheta_l \searrow -\infty} S[\vartheta_l, \vartheta_h] = \infty \quad \text{and} \quad \lim_{\vartheta_h \nearrow +\infty} S[\vartheta_l, \vartheta_h] = \infty, \tag{19}$$

and this result is independent of the fixed argument that is not under the limit. With nonattracting boundaries, the stationary density exists if the speed measure satisfies

$$\lim_{\vartheta_l \searrow -\infty} M[\vartheta_l, \vartheta_h] < \infty \quad \text{and} \quad \lim_{\vartheta_h \nearrow +\infty} M[\vartheta_l, \vartheta_h] < \infty, \tag{20}$$

again independently of the argument that is not under the limit. In our case,

$$s(\vartheta) = \exp\left\{-\int_{\vartheta_0}^{\vartheta} \frac{2\left(\nu^2(\tau) - \nu^1(\tau)\right) + \left(u^2\right)^2 - \left(u^1\right)^2}{\left(u^1 - u^2\right)^2} d\tau\right\}. \tag{21}$$

For the left boundary, assume that in line with condition (i), there exist $\underline{\vartheta} \in \mathbb{R}$ and $\underline{\mu} \geq 0$ such that $\mu_\vartheta(\vartheta) \geq \underline{\mu}$ for all $\vartheta \in (-\infty, \underline{\vartheta})$. Taking $\vartheta_0 = \underline{\vartheta}$, the scale function can be bounded as

$$S[\vartheta_l, \underline{\vartheta}] \geq \int_{\vartheta_l}^{\underline{\vartheta}} \exp\left\{-\int_{\underline{\vartheta}}^{\vartheta} \frac{2\underline{\mu}}{\left(u^1 - u^2\right)^2} d\tau\right\} d\vartheta = \frac{\left(u^1 - u^2\right)^2}{2\underline{\mu}} \left[\exp\left(-\frac{2\underline{\mu}}{\left(u^1 - u^2\right)^2}\left(\vartheta_l - \underline{\vartheta}\right)\right) - 1\right]$$

with the case $\underline{\mu} = 0$ obtained through the limit $\underline{\mu} \searrow 0$. The left limit in (19) (as $\vartheta_l \searrow -\infty$) thus diverges to infinity and the left boundary is nonattracting.

The argument for the right boundary is symmetric. Taking $\bar{\vartheta} \in \mathbb{R}$ and $\bar{\mu} \leq 0$ such that $\mu_\vartheta(\vartheta) \leq \bar{\mu}$ for all $\vartheta \in \left(\bar{\vartheta}, 1\right)$ makes the right limit in (19) hold, and the right boundary is nonattracting.

It turns out that the bounds implied by conditions (20) are marginally tighter. Following the same bounding argument as above, sufficient conditions for (20) to hold are given by the strict inequalities $\underline{\mu} > 0$ and $\bar{\mu} < 0$. Then a unique stationary density for $\vartheta$ (and hence for $\theta$) exists, proving case (a) of the proposition.

The construction also reveals that these bounds are the least tight bounds of this type under which the proposition holds. It is useful to note that the unique stationary density $q(\vartheta)$ is proportional to the speed density $m(\vartheta)$ (and the same holds for $\theta$). Finally, if the limits in Proposition 3.2 do not exist, they can be replaced with appropriate limits inferior and superior.

This discussion has sorted out case (a). Conditions (i') and (ii') are sufficient conditions for the boundaries to be attracting. Lemma 6.1 in Karlin and Taylor (1981) then shows that if the 'attracting' condition is satisfied for a boundary, then $\vartheta$ diverges to this boundary on a set of paths that has a strictly positive probability. This probability is equal to one if the other boundary is non-attracting. Combining these results, we obtain statements (b), (c) and (d). ■

**Proof of Proposition 5.1.** The proof relies on showing that the dynamics of the continuation values $V_t^n$ in the proximity of the boundaries become degenerate in a specific sense. From this fact, I can infer the behavior of the stochastic discount factor implied by the consumption process of the large agent and, consequently, the equilibrium price dynamics.

Without loss of generality, consider the boundary as $\theta \searrow 0$. Using the construction from Duffie and Epstein (1992a), the stochastic discount factor process for the 'large' agent 2 under the subjective probability measure $Q^2$ is given by

$$S_t^2 = \exp\left(-\int_0^t \nu^2(\theta_s)\, ds\right)\left(\frac{Y_t}{Y_0}\right)^{-\gamma}\left(\frac{\zeta^2(\theta_t)}{\zeta^2(\theta_0)}\right)^{-\rho}\left(\frac{\widetilde{J}^2(\theta_t)}{\widetilde{J}^2(\theta_0)}\right)^{\frac{\rho-\gamma}{1-\gamma}}. \tag{22}$$

The proof of Lemma OA.5 in Online Appendix OA.9 shows that $\lim_{\theta\searrow 0}\zeta^2(\theta) = 1$ and $\lim_{\theta\searrow 0}\widetilde{J}^2(\theta) = \bar{V}^2$, implying that $\lim_{\theta\searrow 0}\nu^2(\theta) = \bar{\nu}^2$, with $\bar{V}^2$ and $\bar{\nu}^2$ given in the proof of Lemma OA.2. Homotheticity of preferences implies that individual wealth-consumption ratios are given by

$$\xi^n(\theta) = \frac{1}{\beta}\left(\frac{(1-\gamma)\,\widetilde{J}^n(\theta)}{\zeta^n(\theta)^{1-\gamma}}\right)^{\frac{1-\rho}{1-\gamma}} \qquad n \in \{1,2\} \tag{23}$$

and since $\lim_{\theta\searrow 0}\xi^2(\theta) = \lim_{\theta\searrow 0}\xi(\theta)$, we obtain the limiting behavior of the aggregate wealth-consumption ratio using the above limits for $\zeta^2(\theta)$ and $\widetilde{J}^2(\theta)$.

It remains to be shown that the local drift and volatility in the local evolution of the last two terms of (22) decline to zero as $\theta \searrow 0$, so that

$$d\log S_t^2 \doteq \mu_{S^2}(\theta_t)\, dt + \sigma_{S^2}(\theta_t)\, dW_t = \left[\mu_{S^2}(\theta_t) + u^2\sigma_{S^2}(\theta_t)\right] dt + \sigma_{S^2}(\theta_t)\, dW_t^n$$

with

$$\lim_{\theta\searrow 0}\mu_{S^2}(\theta) = -\bar{\nu}^2 - \gamma\mu_y, \qquad \lim_{\theta\searrow 0}\sigma_{S^2}(\theta) = -\gamma\sigma_y. \tag{24}$$

The results for the limiting behavior of the risk-free rate and return on the aggregate wealth then follow directly from the local behavior of the stochastic discount factor in (24). Details of the calculations proving (24) are provided in Section OA.11 of the Online Appendix. ■

**Proof of Corollary 5.2.** An application of Itô's lemma to $\theta_t = \bar{\lambda}_t^1 / \left(\bar{\lambda}_t^1 + \bar{\lambda}_t^2\right)$ yields

$$d\theta_t = \theta_t(1-\theta_t)\left[\nu_t^2 - \nu_t^1 + \left(\theta_t u^1 + (1-\theta_t)u^2\right)\left(u^2 - u^1\right)\right] dt + \theta_t(1-\theta_t)\left(u^1 - u^2\right) dW_t. \tag{25}$$

The evolution of $\theta$ then implies that for every fixed $t \geq 0$

$$\theta_0 \searrow 0 \qquad \Longrightarrow \qquad \theta_t \to 0,\ P\text{-a.s.}$$

and thus also $\zeta^2(\theta_t) \to 1$ and $\widetilde{J}^2(\theta_t) \to \bar{V}^2$, $P$-a.s.[17] The last two terms in the expression for the stochastic discount factor $S_t^2$, equation (22), converge to one, $P$-a.s., and since $\nu^2(\theta_s)$, $0 \leq s \leq t$ also converges to

[17] This result becomes more transparent if we consider $\zeta^2$ and $\widetilde{J}^2$ as functions of $\vartheta_t = \vartheta(\theta_t)$. The dynamics of $\vartheta_t$ in (10) have bounded drift and volatility coefficients and thus for $\forall\varepsilon > 0$, $\forall k > 0$, it is possible to achieve

$$P[\theta_t < k] = P[\log\theta_t < \log k] > 1 - \varepsilon$$

by setting $\log\theta_0$ sufficiently low.

$\nu^2(0)$ and is bounded, we have $S_t^2 \xrightarrow{P} S_t^2(0)$. Consider a family of random variables $M_t^2 S_t^2(\theta_0)$ indexed by the initial Pareto share $\theta_0$. Since this family is uniformly integrable, then convergence in probability implies convergence in mean, and we obtain the convergence result for bond prices

$$E\left[M_t^2 S_t^2(\theta_0) \mid \mathcal{F}_0\right] \xrightarrow{\theta_0 \searrow 0} E\left[M_t^2 S_t^2(0) \mid \mathcal{F}_0\right].$$

The same argument holds for $M_t^2 S_t^2(\theta_0) Y_t$, which yields the result for the price of individual consumption strips from the aggregate endowment. ■

**Proof of Proposition 5.3.** Agent 1, whose wealth $A^1$ is close to zero, solves

$$\bar{\lambda}_t^1 V_t^1 = \max_{(C^1,\pi^1,\nu^1)} E_t\left[\int_t^\infty \bar{\lambda}_s^1 F\left(C_s^1, \nu_s^1\right) ds\right] \tag{26}$$

subject to (7) and the budget constraint,

$$\begin{aligned}\frac{dA_t^1}{A_t^1} &= \left[r(\theta_t) + \pi_t^1\left([\xi(\theta_t)]^{-1} + \mu_A(\theta_t) + \frac{1}{2}[\sigma_A(\theta_t)]^2 - r(\theta_t)\right) - \frac{C_t^1}{A_t^1}\right] dt + \pi_t^1 \sigma_A(\theta_t) dW_t = \\ &= \mu_{A^1}(\theta_t) dt + \sigma_{A^1}(\theta_t) dW_t\end{aligned} \tag{27}$$

where $\pi^1$ is the portfolio share invested in the risky asset. The solution of this equation determines the consumption-wealth ratio of agent 1 and, consequently, the evolution of her wealth. The local behavior of returns on the risk-free bond $r(\theta)$ and risky asset (14) as $\theta \searrow 0$ is known from Proposition 5.1.

The proof of this proposition is based on showing that the limit of the optimal consumption, portfolio and discount rate choice $\left(C^1, \pi^1, \nu^1\right)$ as $\theta \searrow 0$ is equivalent to taking the limit for equilibrium returns $(r, \xi, \mu_A, \sigma_A)$ as $\theta \searrow 0$ first, and then computing the optimal individual choice of agent 1 under the limiting returns. Details of this proof are provided in Section OA.11 of the Online Appendix. ■

It remains for me to verify that the boundedness assumption for wealth-consumption ratios indeed holds.

**Corollary B.1** *Under parameter restrictions in Assumption A.1, the wealth-consumption ratios are bounded and bounded away from zero.*

**Proof of Corollary B.1.** The critical point is the limits for the consumption-wealth ratios as the Pareto share of one of the agents becomes small. Since the large agent's consumption-wealth ratio converges to that in a homogeneous economy, the relevant parameter restriction is the same as restriction (17) in Assumption A.1. The consumption-wealth ratio of the small agent is derived in the proof of Proposition 5.3. Restriction (18) in Assumption A.1 assures that this quantity is strictly positive, and the wealth-consumption ratio finite.[18] ■

**Proof of Corollary 3.3.** Utilize results in Proposition 5.3 and the fact that $\lim_{\theta\searrow 0} \mu_{A^2}(\theta) = \mu_y$ and $\lim_{\theta\searrow 0} \sigma_{A^2}(\theta) = \sigma_y$, then form the differences in the limiting expected logarithmic growth rates, and compare them to inequalities in Proposition 3.2. ■

**Proof of Proposition 3.4.** The difference in expected logarithmic returns is obtained by computing the

[18] This also confirms that the discount rate restrictions in Assumption OA.3 from Online Appendix OA.9 hold as well, and the proof of Proposition 2.3 presented in Online Appendix OA.9 is now complete.

limiting behavior of

$$\left(\pi^1(\theta) - \pi^2(\theta)\right)\left[[\xi(\theta)]^{-1} + \mu_A(\theta) + \frac{1}{2}(\sigma_A(\theta))^2 - r(\theta)\right] - \frac{1}{2}\left[\left(\pi^1(\theta)\right)^2 - \left(\pi^2(\theta)\right)^2\right](\sigma_A(\theta))^2,$$

utilizing the results for $\theta \searrow 0$ from Propositions 5.1 and 5.3. The first term above is the difference in the risk premium associated with the two portfolios, and the second term is the volatility penalty. The same propositions also contain the results for the consumption-wealth ratios of the two agents. ■

**Proof of Corollary 4.1.** The results are obtained by taking limits of expressions in Proposition 3.4. ■

**Proof of Corollary 4.2.** Assume without loss of generality that $\left|u^2\right| \leq \left|u^1\right|$. The sufficient part is an immediate consequence of Proposition 3.2. Under separable preferences, $\nu^2 - \nu^1 \equiv 0$, and thus if $\left|u^2\right| < \left|u^1\right|$ then conditions (i') and (ii) hold, and agent 2 dominates in the long run under $P$.

For the necessary part, when $u^2 = u^1$, then $\vartheta$ is constant and both agents survive under $P$. When $-u^2 = u^1 = u$, then it follows from inspection of formula (21) in the proof of Proposition 3.2 that conditions (19) are satisfied and the boundaries are non-attracting. Lemma 6.1 in Karlin and Taylor (1981) then implies that both agents survive under $P$. However, even though both agents survive when $-u^2 = u^1$, the speed density $m(\vartheta) \propto 1$ is not integrable on $(-\infty, +\infty)$ and thus there does not exist a finite stationary measure. All above statements are in fact results about the behavior of a Brownian motion with a constant drift.

The result on survival under measure $Q^n$ follows from the fact that the evolution of Brownian motion $W$ under the beliefs of agent $n$ is $dW_t = u^n dt + dW_t^n$. Since the evolution of $\vartheta$ completely describes the dynamics of the economy, substituting this expression into (10) and reorganizing yields the desired result. ■

**Proof of Proposition 4.3.** Since $\theta$ is an Itô diffusion with bounded coefficients and volatility that is bounded away from zero on $(\varepsilon, 1-\varepsilon)$, $\forall \varepsilon > 0$, the stationary density, if it exists, is unique and has a full support. Hence every point in $(0,1)$ is visited with probability one, given any initial condition $\theta_0 \in (0,1)$. ■

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