Robust Preference Expansions*

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Abstract

We propose an approximation method for solving dynamic stochastic general equilibrium models in which agents are concerned about model misspecification. The method relies on a perturbation that treats this robust concern as a first-order concept that is preserved as the volatility of the shocks vanishes. The approximation has a clear economic interpretation and generates solutions with consequences of robust preferences that standard perturbation methods only capture using higher-order terms. In particular, our method generates risk premia in the linear solution and time variation in these risk premia and stochastic volatility effects in the second-order approximation.

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1 Introduction

Perturbation methods are an integral part of the toolkit in macroeconomic dynamic stochastic general equilibrium (DSGE) modeling. These methods rely on the local smoothness of the exact stochastic solution in the neighborhood of a non-stochastic steady state and generate an approximate solution by constructing an expansion of the policy rule, typically in the class of polynomial functions.

The perturbation expansions naturally lead to the notion of the 'order' of expansion, represented by the order of the approximating polynomial. Theoretical results in perturbation analysis specify the conditions under which the approximate solution converges to the exact solution as the order of expansion increases. Also, different orders of expansion used in DSGE modeling lead to particular properties of the approximate solution. Linearization techniques (first-order expansions) are commonly used to capture smooth dynamics of macroeconomic quantities. A second-order expansion is needed to generate nonzero risk premia, and a third-order expansion is necessary to capture the time-variation in these premia.

The notion of convergence as the order of approximation increases is typically not very useful for practical purposes. In nontrivial DSGE models, every additional order of approximation introduces a nontrivial additional computational burden for solution and estimation techniques. It is therefore useful to keep the order of approximation low, while capturing the desired qualitative and quantitative features of the model solution.

In this paper, we derive a solution technique for approximating DSGE models populated by agents endowed with robust and recursive preferences. We utilize the observation that a specific representation of robust utility, the constraint preference specification, exhibits first-order risk aversion that is manifested by kinked indifference curves at certainty. Our approach scales the robust preference parameter jointly with the volatility of the shocks to replicate this feature. This approximation effectively shifts the order of approximation of the preference structure one order lower relative to the rest of the model.

The mixing of the order of approximation may seem to be conceptually inconsistent but we argue that this is not the case. The asymptotic consistency results either capture a situation when the order of approximation increases (e.g., the Taylor's theorem) or the volatility of the underlying exogenous shocks declines to zero (small noise or asymptotic expansion argument). In applications, the DSGE modeler is interested in the performance of the given order of approximation in the stochastic equilibrium, away from the deterministic steady state.

We specifically focus on a second-order approximation and show how to incorporate third-order terms from the preference approximation while retaining the second-order structure. In an example, we show that this modification captures most of the third-order dynamics in the model. Naturally, this result is example-specific but we argue that this result can be expected in a broad class of models where agents are endowed with recursive preferences or concern for robustness.

In a given order of approximation, the error of approximation can be expected to be of higher order (as a function of the perturbation parameter) but the coefficient on the error term can be large. Intuitively, stochastic discount factors for agents endowed with recursive or robust preferences are designed to generate quantitatively relevant asset pricing dynamics and thus are often the primary source of nonlinearities in the model. It is therefore desirable to incorporate the preference structure particularly accurately. Hence our approach, which prioritizes the order of approximation of the preference relation.

As we argued, this approximation takes a very particular stand on the characterization of the preference structure in the vicinity of certainty. But from the perspective of DSGE modeling, this is a mute issue, since we are fitting the model to the dynamics in the stochastic equilibrium. The problem would only become relevant if we had data on economies as the perturbation parameter declines to zero. While this may be important in other economic problems, our goal is to provide a technique with a rigorous foundation that would perform well in the stochastic equilibrium.

1.1 Related literature

Our work builds on a large volume of literature on perturbation methods. Standard techniques, as in Judd (1998), Jin and Judd (2002) or Schmitt-Grohé and Uribe (2004), construct expansions by scaling down the volatility of the exogenous shocks. The approximate solutions lead to globally unstable dynamics when the shocks do not have a bounded support, and Kim et al. (2008) or Andreasen et al. (2013) proposed so-called pruning techniques that augment the law of motion for the state vector in order to stabilize the dynamics of simulated paths. Lan and Meyer-Gohde (2013) construct a perturbation to the nonlinear moving average representation solution that is stable as long as the first-order dynamics is stable.

In order to improve the quality of approximation in nonlinear models in which the ergodic distribution can be far away from the deterministic steady state, some papers implemented methods that shift the expansion point closer to the center of the ergodic distribution. Coeurdacier et al. (2011) consider a 'risky' steady state that takes into account the variance adjustment generated by the curvature of the stochastic discount factor. Alternative approaches involve methods explicitly dealing with heteroskedastic innovations as in Justiniano and Primiceri (2008), Malkhozov and Shamloo (2011), or Benigno et al. (2013).

Our approach is based on the series expansion method of Holmes (1995) and Lombardo (2010). Our specific focus on nonseparable preference structures with concerns for robustness is linked to the risk-sensitive control problems analyzed in James (1992), Campi and James (1996), and Anderson et al. (2012). We however proceed by expanding jointly around the shock volatility parameter and the robustness parameter. In a similar fashion, Kogan and Uppal (2001) expand a separable preference model around unitary risk aversion and Hansen et al. (2007) construct an expansion of the recursive preference model around unitary intertemporal elasticity of substitution.

2 Modeling framework

We consider a dynamic model with a Markov representation

$$x_t = \psi\left(x_{t-1}, w_t\right) \tag{1}$$

where w_t is a $k \times 1$ vector of serially uncorrelated shocks with $w_t \sim N\left(0, I\right)$ and x_t is an $n \times 1$ vector of model variables. The vector x_t incorporates what is often known as exogenous and endogenous state variables as well as control variables. We will later allow for more structure imposed on x_t. The law of motion ψ is assumed to be sufficiently differentiable at the nonstochastic steady state.

Our goal is to solve for an approximation of the law of motion ψ from a set of equilibrium conditions. These equilibrium conditions will have a specific form dictated by the robust preference specification that we utilize and that generalizes the standard assumption of rational expectations. While we motivate the expansion with agents' concern for robustness, observational equivalence with particular specifications of recursive preferences allows us to extend the method to recursive preference structures as well.

2.1 Robust preferences

We have in mind an economy populated by potentially different types of agents who are concerned that the models they use to forecast the dynamics of the economy are potentially misspecified. They treat the model (1) as an approximating or benchmark model and consider stochastic deviations from this model to derive a worst-case scenario that is used as a basis for their decisions. The worst-case model is difficult to distinguish statistically from the approximating model, and the degree of statistical similarity between the two models is controlled by an entropy penalty imposed on the continuation utility. The continuation value for agent $i = 1, \ldots, I$ with a given period utility process u_t^i can be represented as a solution to the minimization problem

$$V_t^i = \min_{M_{t+1}^i} u_t^i + \beta_i \theta_i E_t \left[M_{t+1}^i \log M_{t+1}^i \right] + \beta_i E_t \left[M_{t+1}^i V_{t+1}^i \right] \tag{2}$$

subject to $E_t \left[M_{t+1}^i \right] = 1$. Here, M_{t+1}^i represents the (one-period) distortion of the worst-case model used to evaluate the future relative to the approximating model, and θ_i is the entropy parameter that controls the degree of robustness. As $\theta_i \to \infty$, deviations from the approximating model become prohibitively costly, and we obtain convergence to the rational expectations framework. Hansen and Sargent (2008) provide an extensive treatment of the robust utility problems.

It is well known that the worst-case distortion takes the form

$$M_{t+1}^i = \frac{\exp\left(-\frac{1}{\theta_i} V_{t+1}^i\right)}{E_t \left[\exp\left(-\frac{1}{\theta_i} V_{t+1}^i\right) \right]} \tag{3}$$

and the preference recursion can be written as

$$V_t^i = u_t^i - \beta_i \theta_i \log E_t \left[\exp\left(-\frac{1}{\theta_i} V_{t+1}^i\right) \right]. \tag{4}$$

Taken distortions (3) as given, we can view this model as a model with heterogeneous beliefs. Each type of agents whose concern for robustness is parameterized by θ_i distorts their beliefs by the belief ratio (Radon-Nikodým derivative) M^i given by equation (3). Since the continuation values V^i are determined in equilibrium, the belief heterogeneity arises here endogenously, as different classes of agents fear different states of the world depending on the dynamics of their continuation values.

There exist standard isomorphisms between robust and recursive preference specifications. The worst-case distortion M_{t+1}^i appears in the Euler equations of the robust agent and can be treated as a component of the stochastic discount factor. Consider, for instance, the case $u_t^i = \log\left(C_t^i\right)$ where C^i is the consumption process for agent i. Then the stochastic discount factor takes the form

$$S_{t+1}^i = \beta_i \left(\frac{C_{t+1}^i}{C_t^i} \right)^{-1} \frac{\exp\left(-\frac{1}{\theta_i} V_{t+1}^i\right)}{E_t \left[\exp\left(-\frac{1}{\theta_i} V_{t+1}^i\right) \right]}, \tag{5}$$

which is the standard Epstein and Zin (1989) stochastic discount factor with unitary elasticity of substitution if we interpret $1 + (\theta_i)^{-1}$ as the coefficient of relative risk aversion.

Our main goal is to argue that the nonlinearities introduced through the term M_{t+1}^i

are often of particular quantitative importance, and that it is desirable to treat this term differently in a perturbation approximation. In addition, we provide a theoretically consistent justification for our approach. The approach is based on the scaling of the robustness parameter θ_i jointly with the volatility of the underlying shocks. Given that the parameter θ_i only enters the model through the worst-case distortion M_{t+1}^i, we will separate out the dynamics of M_{t+1}^i from the rest of the model in the next section.

2.1.1 A static example

In order to illustrate our approximation approach, it is useful to introduce a static example. Let the distribution of consumption be given by

$$\log C(\mathsf{q}) = \mu - \frac{1}{2}(\mathsf{q}\sigma)^2 + \mathsf{q}\sigma W$$

where W is a standard normal shock and $\mathsf{q} > 0$ a perturbation parameter that scales the volatility of the shock. The term $\frac{1}{2}(\mathsf{q}\sigma)^2$ assures that $\log E[C(\mathsf{q})] = \mu$ for every q. We want to study agents' preferences over different consumption distributions as the parameter q changes.

We consider three agents who rank consumption distributions under different specifications of preferences. The first agent is endowed with power utility, the other two with two alternative specifications of robust preferences — the multiplier preferences defined in the dynamic case in equation (2), and constraint preferences. Appendix D provides the details for the computations.

1. Power utility

   $$u^{pow} = \frac{1}{1-\gamma}\log E\left[C^{1-\gamma}\right]$$

   where γ is the coefficient of relative risk aversion;

2. Multiplier preferences (a special static case of the preference specification in (2))

   $$u^{mult} = \min_{M,\ E[M]=1} E[M\log C] + \theta E[M\log M] \tag{6}$$

   where θ is the entropy penalty parameter;

3. Constraint preferences (an alternative representation of robust preferences introduced in Hansen and Sargent (2001) and Hansen et al. (2006))

   $$u^{con} = \min_{M,\ E[M]=1} E[M\log C] \qquad \text{s.t. } E[M\log M] \le \eta \tag{7}$$

where η constraints the entropy (or expected log-likelihood ratio) of the worst-case model relative to the approximating model.

The power utility and multiplier preference representations coincide when $\theta = (\gamma - 1)^{-1}$ and the compensation in the average consumption level $\mu(\mathsf{q}) = \log E[C(\mathsf{q})]$ that keeps the agent indifferent between random consumption profiles with volatility $\mathsf{q}\sigma$ satisfies

$$\mu^{pow}(\mathsf{q}) - \mu^{pow}(0_+) = \mu^{mult}(\mathsf{q}) - \mu^{mult}(0_+) = \frac{1}{2}\gamma \mathsf{q}^2\sigma^2 = \frac{1}{2}\frac{1+\theta}{\theta}\mathsf{q}^2\sigma^2. \tag{8}$$

where $\mu(0_+) = \lim_{v\to 0}\mu(v)$. The fact that the average consumption level compensation per unit of volatility declines to zero as volatility decreases,

$$\lim_{\mathsf{q}\to 0}\frac{\mu^{pow}(\mathsf{q}) - \mu^{pow}(0_+)}{\mathsf{q}\sigma} = \lim_{\mathsf{q}\to 0}\frac{1}{2}\frac{1+\theta}{\theta}\mathsf{q}\sigma = 0,$$

reflects the second-order effect of risk aversion on compensation for risk. In the case of multiplier preferences, the worst case model is given by the belief distortion

$$M^{mult} = \frac{\exp\left(-\frac{1}{\theta}\log C\right)}{E\left[\exp\left(-\frac{1}{\theta}\log C\right)\right]} = \exp\left(-\frac{1}{2\theta^2}\mathsf{q}^2\sigma^2 - \frac{1}{\theta}\mathsf{q}\sigma W\right)$$

and as $\mathsf{q}\to 0$, the probability distribution of the worst-case model converges to that of the benchmark model.

On the other hand, the same calculation for the constraint preferences leads to

$$\mu^{con}(\mathsf{q}) - \mu^{con}(0_+) = \sqrt{2\eta}\,\mathsf{q}\sigma + \frac{1}{2}\mathsf{q}^2\sigma^2. \tag{9}$$

and to the corresponding worst-case belief distortion

$$M^{con} = \frac{\exp\left(-\frac{\sqrt{2\eta}}{\mathsf{q}\sigma}\log C\right)}{E\left[\exp\left(-\frac{\sqrt{2\eta}}{\mathsf{q}\sigma}\log C\right)\right]} = \exp\left(-\eta - \sqrt{2\eta}W\right).$$

The first term on the right-hand side of (9) reflects the first-order nature of the compensation for uncertainty built into the constraint preference specification. As $\mathsf{q}\to 0$, the worst-case model distortion does not vanish.

While the multiplier and constraint problems are distinct, it is possible to calibrate them in a way that makes the indistinguishable *locally*. Consider setting two free parameters

$\mu^{con}(0_+)$ and η in the following way:

$$\begin{aligned}\mu^{con}(0_+) &= \mu^{mult}(0_+) - \frac{1}{2}\frac{\sigma^2}{\theta} \\ \eta &= \frac{\sigma^2}{2\theta^2}.\end{aligned} \tag{10}$$

where $\mu^{mult}(0_+)$ can be chosen arbitrarily as a scaling parameter. Then

$$\mu^{con}(\mathsf{q}) = \mu^{mult}(0_+) - \frac{1}{2}\frac{\sigma^2}{\theta} + \frac{\sigma^2}{\theta}\mathsf{q} + \frac{1}{2}\mathsf{q}^2\sigma^2$$

which implies

$$\begin{aligned}\mu^{con}(1) &= \mu^{mult}(0_+) + \frac{1}{2}\frac{1+\theta}{\theta}\sigma^2 = \mu^{mult}(1) \\ \left.\frac{d}{d\mathsf{q}}\mu^{con}(\mathsf{q})\right|_{\mathsf{q}=1} &= \frac{1+\theta}{\theta}\sigma^2 = \left.\frac{d}{d\mathsf{q}}\mu^{con}(\mathsf{q})\right|_{\mathsf{q}=1}\end{aligned}$$

The absolute level of compensation for risk (the 'risk premium') as well as the change in this compensation and the quantity of aggregate risk changes (the 'price of risk') are thus identical for the two preference structures in the stochastic equilibrium when $\mathsf{q} = 1$. Naturally, these compensations will differ at different values of q, but it is not clear a priori which of the two preference structures should be preferred, as we have no macroeconomic data on economies for $\mathsf{q} \neq 1$.

In other words, the entropy penalty parameter θ and the entropy constraint parameter η do not have a direct structural interpretation. For instance, Anderson et al. (2000) and others advocate the use of detection error probabilities as a way of disciplining the extent of the concern of robustness. Holding the detection error probability constant will, however, imply a parameter θ that depends on the characteristics of the stochastic environment.

Figure 1 compares the compensation for risk for the case of multiplier and constraint preferences as a function of the perturbation parameter q. The parameters are chosen in line with (10) so that the implications for the risk premium coincide in the stochastic equilibrium ($\mathsf{q} = 1$). The dotted lines represent the implications for the risk premium of a linear approximation of the two preference structures. While the linear approximation of the multiplier preferences does not generate any risk premia, the situation is different for the case of constraint preferences. We will show in the example in Section 6 that this logic also extends to higher orders of approximation. In particular, our second-order approximation will generate asset pricing implications which are only present in the conventional perturbation approximation of at least third order.

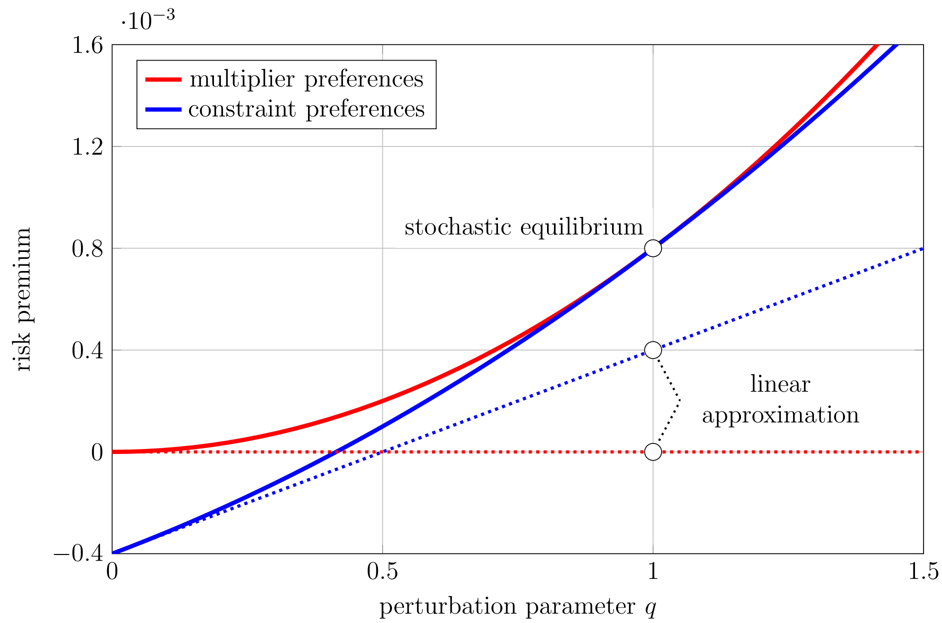

Figure 1: Risk premium as a function of the perturbation parameter q for the multiplier and constraint preference models, parameterized by $\theta = 1$, $\sigma = 0.02$. The dotted lines represent linear approximations of the risk premium function around $q = 0$. Circles capture the risk premium evaluated at the stochastic equilibrium ($q = 1$) and at the linear approximation around $q = 0$ of the stochastic equilibrium under the multiplier and constraint preferences.

From the comparison of the results for multiplier and constraint preferences it follows that one possible way of choosing θ is to keep the entropy of the worst-case model relative to the benchmark model (roughly) constant as $\mathsf{q} \to 0$. Utilizing equations (8) and (9) we deduce that θ should be scaled by q, which leads to the following preference representation

$$u^{mult} = \min_{M,\ E[M]=1} E\left[M \log C\right] + \mathsf{q}\theta E\left[M \log M\right]$$

We will utilize this scaling in our series expansion method below. Scaling the entropy penalty by q effectively generates first-order uncertainty aversion effects reflected in a kink in the indifference curves around certainty. The purpose of this paper is to incorporate these first-order effects into the approximation techniques based on series expansion methods in a rigorous and tractable way.

2.2 The model

The endogenous law of motion ψ in equation (1) is unknown and needs to be solved for from a set of equilibrium conditions. We assume that we can represent these equilibrium conditions in the form

$$0 = E_t\left[\tilde{g}\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right] \tag{11}$$

where $\tilde{g}$ is an $n \times 1$ vector function and the dynamics for x_t is implied by (1). These conditions include, with sufficient generality, expectational and nonexpectational equations, including laws of motion for exogenous variables. There are well-known saddle-point stability conditions on the system (11) that lead to a unique equilibrium of the linear approximation (see Blanchard and Kahn (1980) or Sims (2002)) and we assume that these are satisfied.

The equilibrium conditions (11) include the Euler equations of different agents in the model. We want to allow for heterogeneity in agents' concerns for robustness which implies that the model can involve multiple different M_{t+1}^{i}. We achieve substantial generality by assuming that we can write the j-th component of $\tilde{g}$ as

$$\tilde{g}^{j}\left(x_{t+1}, x_t, x_{t-1}, z_{t+1}, z_t\right) = M_{t+1}^{\sigma_j} g^{j}\left(x_{t+1}, x_t, x_{t-1}, z_{t+1}, z_t\right).$$

where $\sigma_j \in \{0, 1, \ldots, I\}$ indexes the agent whose concern for robustness is associated with the distortion of the j-th equation, with $M_{t+1}^{0} \equiv 1$. In particular, all nonexpectational equations and all equations not involving agents' preferences will have $\sigma_j = 0$. System (11) can then be written as

$$0 = E_t\left[\mathbb{M}_{t+1} g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right] \tag{12}$$

where $\mathbb{M}_{t+1} = \operatorname{diag}\left\{M_{t+1}^{\sigma_1}, \ldots, M_{t+1}^{\sigma_n}\right\}$ is a diagonal matrix of the belief distortions, and g is independent of the robustness parameters θ_i.

3 Series expansions

We utilize the series expansion method (see Holmes (1995)) which has been widely used in applied mathematics to derive approximate solutions to differential equations.[1] Lombardo (2010) implemented the series expansion method to construct perturbation approximations to recursive DSGE models. This solution technique generates a recursively linear solution for the individual orders of approximation. The dynamics of higher-order approximations constructed using the series expansion method are globally stable as long as the first-order dynamics are stable. This is a significant advantage over standard perturbation techniques (Schmitt-Grohé and Uribe (2004)) where second- and higher-order dynamics are by construction globally unstable, and one has to rely on additional methods to deal with this instability.[2]

[1] McQuade (2013) provides a recent application of this method in the asset pricing literature.

[2] Kim et al. (2008) propose a 'pruning' technique which drops higher-order terms generates by iterations on the law of motion for the state vector. Andreasen et al. (2013) provide a more systematic approach to the pruning of higher-order solutions.

We will discuss an explicit solution for the second-order approximation. This second-order approximation features prominently in our asset pricing applications. We first show how to expand all quantities that are of our interest and then, in Section 4, we incorporate these expansions into the approximation of the set of equilibrium conditions that describe our dynamic model.

Consider a class of models indexed by the perturbation parameter q:

$$x_t(\mathsf{q}) = \psi\left(x_{t-1}(\mathsf{q}), \mathsf{q}w_t, \mathsf{q}\right) \tag{13}$$

and assume that there exists a series expansion of x_t around $\mathsf{q} = 0$:

$$x_t = x_{0t} + \mathsf{q}x_{1t} + \frac{1}{2}\mathsf{q}^2 x_{2t} + \ldots.$$

The processes x_{jt}, $j = 0, 1, \ldots$ can be viewed as derivatives of x_t with respect to the perturbation parameter, and their laws of motion can be inferred by differentiating (13) j times and evaluating the derivatives at $\mathsf{q} = 0$:

$$\begin{aligned} x_{0t} &= \psi\left(x_{0t-1}, 0, 0\right) \\ x_{1t} &= \psi_x x_{1t-1} + \psi_w w_t + \psi_q \\ x_{2t} &= \psi_x x_{2t-1} + \psi_{xx}\left(x_{1t-1} \otimes x_{1t-1}\right) + 2\psi_{xw}\left(x_{1t-1} \otimes w_t\right) + 2\psi_{xq} x_{1t-1} + \\ &\quad + \psi_{ww}\left(w_t \otimes w_t\right) + 2\psi_{wq} w_t + \psi_{qq}. \end{aligned} \tag{14}$$

Appendix A provides details on the derivation and tensor algebra.

Observe that the expansion has a recursively linear structure, an inherent feature of the series expansion method. The law of motion for x_{0t} is deterministic and in stationary models, $x_{0t} = \bar{x}_0$ is constant. The dynamics for x_{2t} is nonlinear only in x_{1t}. Therefore, stable dynamics for x_{1t} also implies stable dynamics for x_{2t} (and this is also true for higher-order terms). Contrary to standard perturbation methods, series expansions of all orders will generate stable dynamics as long as ψ_x is a stable matrix.

In our solution, the solution for ψ will explicitly depend on q whenever there are agents in the model who have concerns for robustness. It is well known that in standard rational-expectations perturbation solutions (see, for instance, Schmitt-Grohé and Uribe (2004)), the partial derivatives of ψ with respect to q are zero. This is not the case for the robust approximation.

3.1 Distortions

Building on the robust control literature, Borovička and Hansen (2013) propose an expansion of robust preferences that preserves a nontrivial worst-case distortion even as $\mathsf{q} \to 0$. This expansion scales the robustness parameter θ jointly with the volatility of the shocks.

The task is to construct such an expansion of the recursion for the continuation values V_t^i and for the belief distortions M_t^i. We will start with the latter. We want to focus on the second-order expansion of the worst-case distortion and in order to do so, we will use a third-order expansion of the continuation value.

Assume there exists a series expansion

$$V_{t+1}^i \approx V_{0,t+1}^i + \mathsf{q} V_{1,t+1}^i + \frac{\mathsf{q}^2}{2} V_{2,t+1}^i + \frac{\mathsf{q}^3}{6} V_{3,t+1}^i. \tag{15}$$

Our intention is to construct the expansion

$$M_{t+1}^i = \frac{\exp\left(-\frac{1}{\mathsf{q}\theta_i} V_{t+1}^i(\mathsf{q})\right)}{E_t\left[\exp\left(-\frac{1}{\mathsf{q}\theta_i} V_{t+1}^i(\mathsf{q})\right)\right]} \approx M_{0,t+1}^i + \mathsf{q} M_{1,t+1}^i + \frac{\mathsf{q}^2}{2} M_{2,t+1}^i.$$

Substituting in expression (15) and noting that V_{0t+1}^i is a deterministic term, we can approximate M_{t+1}^i with

$$M_{t+1}^i \approx \frac{\exp\left(-\frac{1}{\theta_i}\left(V_{1,t+1}^i + \frac{\mathsf{q}}{2} V_{2,t+1}^i + \frac{\mathsf{q}^2}{6} V_{3,t+1}^i\right)\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}\left(V_{1,t+1}^i + \frac{\mathsf{q}}{2} V_{2,t+1}^i + \frac{\mathsf{q}^2}{6} V_{3,t+1}^i\right)\right)\right]}$$

Differentiating with respect to q and evaluating at $\mathsf{q} = 0$, we obtain the expansion

$$\begin{aligned}
M_{0t+1}^i &= \frac{\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)\right]} \\
M_{1t+1}^i &= -\frac{1}{2\theta_i} M_{0t+1}^i \left[V_{2,t+1}^i - E_t\left[M_{0t+1}^i V_{2,t+1}^i\right]\right] \\
M_{2,t+1}^i &= -\frac{1}{2\theta_i}\left[M_{1,t+1}^i V_{2,t+1}^i - M_{0,t+1}^i E_t\left[M_{1,t+1}^i V_{2,t+1}^i\right]\right] - \\
&\quad -\frac{1}{3\theta}\left[M_{0,t+1}^i V_{3,t+1}^i - M_{0,t+1}^i E_t\left[M_{0,t+1}^i V_{3,t+1}^i\right]\right] + \\
&\quad +\frac{1}{2\theta} M_{1,t+1}^i E_t\left[M_{0,t+1}^i V_{2,t+1}^i\right].
\end{aligned} \tag{16}$$

This expansion is distinctly different from the standard polynomial expansion familiar from the perturbation literature. First, observe that M^i_{0t+1} is not constant, as one would expect for a zeroth-order term, but rather nonlinear in V^i_{1t+1}. However, observe that $E_t\left[M^i_{0t+1}\right] = 1$, and we can thus treat M^i_{0t+1} as a change of measure that will adjust the distribution of shocks that are correlated with M^i_{0t+1}. We will show that with Gaussian shocks, we can still preserve tractability.

Further notice that $E_t\left[M^i_{1t+1}\right] = E_t\left[M^i_{2t+1}\right] = 0$. This has important implications for the expansion of the product $\mathbb{M}_{t+1}g_{t+1}$ in expression (12). Since g_{0t+1} is deterministic, we obtain

$$E_t\left[\mathbb{M}_{1t+1}g_{0t+1}\right] = E_t\left[\mathbb{M}_{2t+1}g_{0t+1}\right] = 0.$$

A second-order expansion of (12) will therefore not contain the $M^i_{2,t+1}$ term and we do not need to expand the continuation values to the third order.

This representation also reveals which terms will contribute to the individual orders of expansion. In addition to the terms obtained in the standard perturbation method, the first-order expansion will feature the term $E_t\left[\mathbb{M}_{0t+1}g_{1t+1}\right]$ where $\mathbb{M}_0$ captures the first-order dynamics of the continuation values V^i, and the second-order expansion will contain terms $E_t\left[\mathbb{M}_{0,t+1}g_{2,t+1}\right]$ and $E_t\left[\mathbb{M}_{1,t+1}g_{1,t+1}\right]$.

Our proposed modification of the perturbation method thus accentuates the role of the worst-case distortions. A similar approach has been used by Ilut and Schneider (2011) and Bianchi et al. (2013) where the dynamics of the worst-case model plays a role in the first-order dynamics of the model. The approach taken in these papers is based on the multiple priors preferences of Gilboa and Schmeidler (1989) generalized to a recursive framework by Epstein and Schneider (2003) and allows fo substantially more flexibility in specifying the belief distortions of the individual shocks. The authors overcome this problem by disciplining the belief distortions by data. On the other hand, our approach provides a tight restriction on the worst-case distortion of the joint distribution of shocks, with a single degree of freedom given by the penalty parameter θ.

3.2 Continuation values

We now turn to the approximation of continuation values. Recursion (4) can be written as

$$V^i_t\left(\mathsf{q}\right) = u^i_t\left(\mathsf{q}\right) - \beta_i\left(\mathsf{q}\theta_i\right)\log E_t\left[\exp\left(-\frac{1}{\mathsf{q}\theta_i}V^i_{t+1}\left(\mathsf{q}\right)\right)\right] \tag{17}$$

We want to construct the first- and second-order approximation of (17) that we can use in the expansion of the belief distortions M^i_{t+1} derived in Section 3.1. We are looking for the approximation in the form

$$V_t^i(\mathsf{q}) \approx V_{0t}^i + \mathsf{q} V_{1t}^i + \frac{\mathsf{q}^2}{2} V_{2t}^i.$$

The zero-th order approximation is nonstochastic and can be found immmediately by setting $\mathsf{q} = 0$:

$$V_{0t}^i = (1-\beta_i)^{-1} u_{0t}^i. \tag{18}$$

Higher-order terms in the expansions are derived by sucessive differentiation with respect to q and are given by the recursions

$$V_{1t}^i = u_{1t}^i - \beta_i \theta_i \ln E_t \left[\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i \right) \right] \tag{19}$$

and

$$V_{2t}^i = u_{2t}^i + \beta E_t \left[\frac{\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)\right]} V_{2t+1}^i \right] = u_{2t}^i + \beta_i \tilde{E}_t^i \left[V_{2t+1}^i \right] \tag{20}$$

where the $\tilde{E}_t^i[\cdot]$ expectation is under the distorting martingale

$$M_{0t+1}^i = \frac{\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i} V_{1t+1}^i\right)\right]}.$$

We consider period utility functions of the type $u_t^i = u^i(x_t)$.[3] With this assumption, it is natural to expect a solution for the continuation values in the form $V_t^i = V^i(x_t)$. Expanding $u_t^i(\mathsf{q}) = u^i(x_t(\mathsf{q}), \mathsf{q})$ and $V_t^i(\mathsf{q}) = V^i(x_t(\mathsf{q}), \mathsf{q})$, and using the method of undetermined coefficients in recursions (19) and (20), we obtain a set of equations for the partial derivatives of V^i in the solutions for V_{1t}^i and V_{2t}^i,

$$\begin{aligned} V_{1t}^i &= V_x^i x_{1t} + V_q^i \\ V_{2t}^i &= V_x^i x_{2t} + V_{xx}^i (x_{1t} \otimes x_{1t}) + 2 V_{xq}^i x_{1t} + V_{qq}^i \end{aligned} \tag{21}$$

where the coefficients can be derived from recursion (19) and (20) using the method of undetermined coefficients. These coefficients depend on the (still unknown) derivatives of ψ but we show below how to solve for all these coefficients jointly. Details of the computations are provided in Appendix A.

The linear structure of V_{1t}^i also has an important implication for the worst-case distortion

[3]Dependence on lagged values of economic variables such as in the case of habit formation models can be treated by appropriately augmenting the vector x_t with lags of these variables.

constructed from M^i_{0t+1}. Substituting into (16) yields

$$M^i_{0t+1} = \frac{\exp\left(-\frac{1}{\theta_i} V^i_x \psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i} V^i_x \psi_w w_{t+1}\right)\right]}. \tag{22}$$

This implies that for a function $f(w_{t+1})$ with a shock vector $w_{t+1} \sim N(0, I)$,

$$E_t\left[M^i_{0t+1} f(w_{t+1})\right] = \tilde{E}^i_t \left[f(w_{t+1})\right] \tag{23}$$

where, under the $\tilde{\cdot}^i$ measure, the vector w_{t+1} has the following distribution:

$$w_{t+1} \sim N\left(-\frac{1}{\theta_i}\left(V^i_x \psi_w\right)', I_k\right).$$

4 Model solution

With the expansions at hand, we can construct the approximation of the model (12) and derive its solution. The nonlinearities in the expansion of $\mathbb{M}_{t+1}$ motivate an iterative procedure.

We will proceed as follows. Our intention is to expand the system with the function

$$g_{t+1} = g\left(x_{t+1}(\mathsf{q}), x_t(\mathsf{q}), x_{t-1}(\mathsf{q}), \mathsf{q}w_{t+1}, \mathsf{q}w_t, \mathsf{q}\right) = g_{0t+1} + \mathsf{q} g_{1t+1} + \frac{\mathsf{q}^2}{2} g_{2t+1}. \tag{24}$$

The zeroth-order approximation of the system is

$$0 = E_t\left[\mathbb{M}_{0t+1} g_{0t+1}\right]$$

Since g_{0t+1} is assumed to be deterministic and $E_t\left[\mathbb{M}_{0t+1}\right] = 1$, this equation dictates $g_{0t+1} = 0$. This amounts to solving for the deterministic path (typically the deterministic steady state) of the model

$$0 = g\left(x_{0t+1}, x_{0t}, x_{0t-1}, 0, 0\right).$$

4.1 First-order expansion

The first-order expansion of the system (12) is

$$0 = E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right] + E_t\left[\mathbb{M}_{1t+1} g_{0t+1}\right] \tag{25}$$

where the last term on the right-hand side is zero because $E_t[\mathbb{M}_{1t+1}] = 0$. We will use this equation to solve for the coefficient matrices in the law of motion

$$x_{1t} = \psi_x x_{1t-1} + \psi_w w_t + \psi_q. \tag{26}$$

The term $\mathbb{M}_{0t+1}$ implies that the first equation is not a linear one — it depends nonlinearly on the continuation values V^i_{1t+1}, which in turn depend on ψ_w (see equation (22)). However, observe that the first-order expansion of (24) takes a linear form

$$g_{1t+1} = g_{x+} x_{1t+1} + g_x x_{1t} + g_{x-} x_{1t-1} + g_{w+} w_{t+1} + g_w w_t + g_q$$

Substituting in for x_{1t+1} using its linear low of motion, we see that $\mathbb{M}_{0t+1}$ will only have an impact on the term $(g_{x+}\psi_w + g_{w+})\, w_{t+1}$. Under rational expectations, this term has a zero mean. Under the beliefs distorted by the zero-th order contribution of the concern for robustness $\mathbb{M}_{0t+1}$, this term will contribute a constant to equation (25).

This observation suggests we proceed as follow. Substituting into equation (25), we can write it in terms of x_{1t-1}, w_t and w_{t+1}. The coefficients on x_{t-1} and w_t imply a pair of equations

$$\begin{aligned} 0 &= (g_{x+}\psi_x + g_x)\,\psi_x + g_{x-} \\ 0 &= (g_{x+}\psi_x + g_x)\,\psi_w + g_w \end{aligned}$$

which can be solved for ψ_x and ψ_w using standard methods. These coefficient matrices will not differ from those obtained in the rational expectations model. With ψ_w, we can reconstruct $\mathbb{M}_{0t+1}$ using the procedure from Section 3.2, and then solve for ψ_q. Define

$$\tilde{\mathbb{E}}_t[w_{t+1}] = \left[\tilde{E}^{\sigma_1}_t[w_{t+1}] \;\; \ldots \tilde{E}^{\sigma_n}_t[w_{t+1}]\right] = \left[E_t\left[M^{\sigma_1}_{0t+1} w_{t+1}\right] \;\; \ldots \;\; E_t\left[M^{\sigma_1}_{0t+1} w_{t+1}\right] \;\right]$$

the matrix consisting of columns that represent conditional means of w_{t+1} under distortions associated with individual equations of the set of equilibrium conditions (25). Then the constant term in (25) implies a condition

$$0 = (g_{x+}\psi_x + g_{x+} + g_x)\,\psi_q + \left[g_q + \mathrm{diag}\left((g_{x+}\psi_w + g_{w+})\,\tilde{\mathbb{E}}_t[w_{t+1}]\right)\right]$$

where the diag $(\cdot)$ operator generates a column vector from the diagonal of a matrix.

In sum, the concern for robustness contributes in the first-order dynamics with a constant term ψ_q to the policy rule (26). ψ_q will be zero in a rational expectations model where $\tilde{\mathbb{E}}_t[w_{t+1}] = 0$ and where g_{t+1} does not explicitly depend on q. We will later show how the drift term changes when the law of motion is represented under the agents' worst-case models.

This drift term also generates risk premia on cash flows which are exposed to shocks which the agent fears. Consider a cash flow with payoff $c'w_{t+1}$. The expected payoff of this cash flow is zero but under agent's i worst case model, this is

$$\tilde{E}_t^i \left[c'w_{t+1}\right] = -\frac{1}{\theta_i} V_x^i \psi_w c$$

which reflects the covariance of the cash flow with the robust adjustment component of the stochastic discount factor (5). Although agents behave as risk-neutral in the first-order approximation, from the perspective of a rational expectations observer they nevertheless require a compensation for their pessimistic beliefs about these cash flows.

4.2 Second-order expansion

Given the more involved algebra of the solution, we only outline a sketch here, with details deferred to Appendix C. The second-order expansion of the equilibrium conditions is given by

$$0 = E_t \left[\mathbb{M}_{0t+1} g_{2t+1}\right] + 2E_t \left[\mathbb{M}_{1t+1} g_{1t+1}\right]. \tag{27}$$

The first term on the right-hand side is a standard second-order expansion with constant mean distortions of the shocks in g_{2t+1} imposed by the known (from the solution of the first-order approximation) worst-case distortion $\mathbb{M}_{0t+1}$. This term will lead to a standard second-order solution, represented by systems of linear equations for the unknown second-order derivatives of ψ.

The second term has a different structure. It contains the term g_{1t+1}, which will only depend on the known first-order dynamics of the law of motion ψ, and an unknown matrix $\mathbb{M}_{1t+1}$ constructed from distortions

$$M_{1t+1}^i = -\frac{1}{2\theta_i} M_{0t+1}^i \left[V_{2,t+1}^i - E_t \left[M_{0t+1}^i V_{2,t+1}^i\right]\right]$$

The term M_{1t+1}^i is the scaled innovation of V_{2t+1}^i under the worst-case measure. Given the quadratic structure implied for the second-order dynamics, we can write

$$M_{1t+1}^i = -\frac{1}{2\theta_i} M_{0t+1}^i \left(A^i \left(x_{1t}\right) \left(w_{t+1} - \tilde{E}_t^i \left[w_{t+1}\right]\right) + B^i \left(w_{t+1} \otimes w_{t+1} - \tilde{E}_t^i \left[w_{t+1} \otimes w_{t+1}\right]\right)\right) \tag{28}$$

where $A^i \left(x_{1t}\right)$ is a row vector that is a linear function of x_{1t} and B^i is a constant row vector. Both $A^i \left(x_{1t}\right)$ and B^i depend on the second-order derivatives of V^i and ψ. Further observe that we can write

$$2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right] = 2E_t\left[\mathbb{M}_{1t+1}\left(g_{x+}\psi_w + g_{w+}\right)w_{t+1}\right]$$

and thus, given the second-order expansion for V^i, the term $E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]$ can be solved in closed form, since it only contains moments of w_{t+1} under the worst-case measure distorted by M^i_{0t+1}.

Given the above considerations, we can express the system (27) as a function of x_{1t-1}, x_{2t-1}, w_t and products of these terms, and use the method of undetermined coefficients to solve for the unknown second-derivative matrices of ψ from the low of motion

$$\begin{aligned} x_{2t} &= \psi_x x_{2t-1} + \psi_{xx}\left(x_{1t-1}\otimes x_{1t-1}\right) + 2\psi_{xw}\left(x_{1t-1}\otimes w_t\right) + 2\psi_{xq}x_{1t-1} + \\ &\quad + \psi_{ww}\left(w_t\otimes w_t\right) + 2\psi_{wq}w_t + \psi_{qq}. \end{aligned}$$

Although these equations are linear in the second derivatives of ψ, they also depend on the second derivatives of the continuation values V^i derived in Section 3.2, which depend on the derivatives of ψ as well. However, we show in Appendix C that there is an ordering in which both the second derivatives of V^i and ψ can be solved for sequentially, without a need for an iterative procedure.

5 Approximating and worst-case dynamics

The approximate dynamics derived in the preceding section with $w_{t+1} \sim N\left(0, I_k\right)$ is the solution of the economic model that incorporates agents' concern for robustness but, at the same time, is represented under the approximating model, which is typically associated with the data generating process.

At the same time, it is possible to represent the law of motion for the economy under the worst-case model perceived by the economic agent. This is useful when we want to derive dynamic responses of the model under the agent's subjective beliefs.

In order to do so, we will proceed somewhat differently than in Section 3.1. Although M^i_{0t+1} is a strictly positive term with a unitary mean that can be used as a change of measure, the higher-order terms are not guaranteed to be strictly positive. An alternative is to use a second-order expansion of the value function in the non-expanded expresssion for the change of measure:

$$M^i_{t+1} \approx \hat{M}^i_{t+1} \equiv \frac{\exp\left(-\frac{1}{\theta_i}\left(V^i_{1t+1} + \frac{1}{2}V^i_{2t+1}\right)\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}\left(V^i_{1t+1} + \frac{1}{2}V^i_{2t+1}\right)\right)\right]}$$

This expression is strictly positive and has a unitary mean.[4] Also, since V^i_{1t+1} is linear in w_{t+1} and V^i_{2t+1} is quadratic, we can write it as

$$\hat{M}^i_{t+1} = \frac{\exp\left(\left(\hat{A}^i_0 + \hat{A}^i_1 x_{1t}\right)' w_{t+1} + \hat{B}^i \left(w_{t+1} \otimes w_{t+1}\right)\right)}{E_t\left[\exp\left(\left(\hat{A}^i_0 + \hat{A}^i_1 x_{1t}\right)' w_{t+1} + \hat{B}^i \left(w_{t+1} \otimes w_{t+1}\right)\right)\right]}$$

and, utilizing formula (33), we deduce that under the distorted measure $\dot{\cdot}^i$, the shock w_{t+1} is distributed as $w_{t+1} \sim N\left(\hat{\mu}^i_0 + \hat{\mu}^i_1 x_{1t}, \hat{\sigma}^i \left(\hat{\sigma}^i\right)'\right)$ with

$$\begin{aligned} \hat{\sigma}^i \left(\hat{\sigma}^i\right)' &= \left(I_k - \operatorname{sym}\left[\operatorname{mat}_{k,k}\left(2\hat{B}^i\right)\right]\right)^{-1} && (29) \\ \hat{\mu}^i_0 + \hat{\mu}^i_1 x_{1t} &= \hat{\sigma}^i \left(\hat{\sigma}^i\right)' \left(\hat{A}^i_0 + \hat{A}^i_1 x_{1t}\right) && (30) \end{aligned}$$

The approximate distortion $\hat{M}^i_{t+1}$ therefore induces a time-varying change in the drift of the shock that is a linear function of the state vector x_{1t}, and a constant adjustment in its volatility. Further, we can write $w_{t+1} = \hat{\mu}^i_0 + \hat{\mu}^i_1 x_{1t} + \hat{\sigma}^i \hat{w}_{t+1}$ where $\hat{w}_{t+1}$ is a multivariate standard normal shock under $\dot{\cdot}^i$, which implies the following dynamics for the solution of the model under the worst-case beliefs of agent i:

$$\begin{aligned} x_{0t} &= \psi\left(x_{0t-1}, 0\right) && (31) \\ x_{1t} &= \psi_x x_{1t-1} + \psi_w \hat{w}^i_t + \psi_w \hat{\mu}^i_0 + \psi_q \\ x_{2t} &= \psi_x x_{2t-1} + \left[\psi_{xx} + 2\psi_{xw}\left(I_n \otimes \hat{\mu}^i_1\right) + \psi_{ww}\left(\hat{\mu}^i_1 \otimes \hat{\mu}^i_1\right)\right]\left(x_{1t-1} \otimes x_{1t-1}\right) + \\ &\quad + \left[2\psi_{xw}\left(I_n \otimes \hat{\sigma}^i\right) + \psi_{ww}\left(\left(\hat{\mu}^i_1\right) \otimes \hat{\sigma}^i + \left[\hat{\sigma}^i \otimes \left(\hat{\mu}^i_1\right)_{\cdot j}\right]^k_{i=1}\right)\right]\left(x_{1t-1} \otimes \hat{w}^i_t\right) + \\ &\quad + \left[2\psi_{xq} + 2\psi_{xw}\left(I_n \otimes \hat{\mu}^i_0\right) + 2\psi_{wq}\hat{\mu}^i_1 + \psi_{ww}\left(\hat{\mu}^i_0 \otimes \hat{\mu}^i_1 + \hat{\mu}^i_1 \otimes \hat{\mu}^i_0\right)\right] x_{1t-1} + \\ &\quad + \psi_{ww}\left(\hat{\sigma}^i \otimes \hat{\sigma}^i\right)\left(\hat{w}^i_t \otimes \hat{w}^i_t\right) + \left[2\psi_{wq}\hat{\sigma}^i + \psi_{ww}\left(\hat{\mu}^i_0 \otimes \hat{\sigma}^i + \hat{\sigma}^i \otimes \hat{\mu}^i_0\right)\right]\hat{w}^i_t + \\ &\quad + \psi_{qq} + \psi_{ww}\left(\hat{\mu}^i_0 \otimes \hat{\mu}^i_0\right) + 2\psi_{wq}\hat{\mu}^i_0 \end{aligned}$$

The change of measure thus generates a rather sophisticated nonlinear impact on the dynamics of the model. First, observe that the first-order dynamics feared by the agent need not be stationary even if the approximating model is, as the autoregressive coefficient changes from ψ_x to $\psi_x + \psi_w \hat{\mu}^i_1$. The autoregressive coefficient of the second derivative x_{2t} is not impacted

[4] Moreover, this expression corresponds to the first-order logarithmic expansion of M^i_{t+1}, compensated to make its mean equal to one. We choose this first-order expansion as an appropriate one here, since the second-order term in the expansion of M^i_{t+1} does not contribute to the solution of the second-order approximation of the model.

in this second-order approximation but the impact of the first-order terms on x_{2t} changes. Also, although the volatility of the shock w_{t+1} is only distorted by a constant transformation (29), the implied change in the conditional volatility of the model is more complex because of the 'stochastic volatility' term $x_{1t-1} \otimes \hat{w}_t^i$, which is now scaled by $2\psi_{xw}\left(I_n \otimes \hat{\sigma}^i\right)$.

5.1 Shock elasticities and nonlinear impulse response functions

The macroeconomics literature uses impulse response functions as a standard tool to evaluate the dynamics of equilibrium models. In a (log)linear VAR framework, the impulse responses are functions of the horizons of the response but are independent of the state at the time of the impact of the shock, the future shock distribution and the magnitude and the sign of the initial shock. In a nonlinear framework, impulse responses generally depend on all these features of the model, and one has to be specific about the type of impulse response experiment conducted in the model evaluation exercise, see Gallant et al. (1993), Koop et al. (1996), or Gourieroux and Jasiak (2005).

Borovička and Hansen (2013) propose a related method that closely links the macroeconomics and asset pricing literatures. The experiment is to compute the sensitivity of expected cash flows and expected returns associated with these cash flows to marginal changes in the exposure of the cash flows to economic shocks. Consider again the model dynamics (1)

$$x_t = \psi\left(x_{t-1}, w_t\right)$$

This model will typically be assumed to be stationary. However, our objects of interest — cash flows and stochastic discount factors — are inherently nonstationary and grow or decay over time. We therefore use the model for x_t as a way to model stationary increments of processes called additive functionals:

$$Y_t = Y_0 + \sum_{s=0}^{t-1} \kappa\left(x_s, w_{s+1}\right).$$

A frequent example of the increment to the additive functional is

$$\kappa\left(x_s, w_{s+1}\right) = \beta\left(x_s\right) + \alpha\left(x_s\right) \cdot w_{s+1}$$

where the state-dependence of $\alpha\left(x_s\right)$ represents a model of stochastic volatility. Examples of additive functionals include logarithms of cash flows and stochastic discount factors. Because asset pricing modeling requires to compute conditional expectations of *levels* of variables rather than logs, we also define a multiplicative functional $M_t = \exp\left(Y_t\right)$.

A typical impulse response experiment considers an impulse which consists of a mean shift in the distribution of the initial shock. We proceed differently and define a perturbation in the direction $\alpha_h(x_0)$ as

$$\log H_1(\mathsf{r}) = \mathsf{r}\alpha_h(x_0) \cdot w_1 - \frac{1}{2}\mathsf{r}^2 \left|\alpha_h(x_0)\right|^2$$

where r is an auxiliary parameter and the direction vector is normalized so that $E\left[\left|\alpha_h(x_0)\right|^2\right] = 1$. We use $H_1(\mathsf{r})$ to perturb the dynamics of the original multiplicative functional M_t in period 1. Specifically, we construct the perturbed dynamics

$$\log M_t + \log H_1(\mathsf{r})$$

and compare the expectation of M_t with the expecattion of $M_t H_1(\mathsf{r})$ for small perturbations by computing the derivative

$$\varepsilon_m(x,t) = \frac{d}{d\mathsf{r}} \log E\left[M_t H_1(\mathsf{r}) \mid x_0 = x\right]\bigg|_{\mathsf{r}=0}.$$

The state-dependent function $\varepsilon_m(x,t)$ is the shock elasticitiy function for the multiplicative functional M_t.

Observe that $H_1(\mathsf{r})$ has mean one, so that perturbing M_t by $H_1(\mathsf{r})$ does not produce any direct mean shifts through the mean of $H_1(\mathsf{r})$. The effect on the conditional expectation, captured by the shock elasticity function, rather comes from marginally increasing the volatility (exposure) of M_t through $H_1(\mathsf{r})$ in the direction of shocks that is captured by $\alpha_h(x_0)$.

This experiment corresponds more to the tradition of the asset pricing literature that compares risk premia on assets with different exposures to risk. However, it is straightforward to show that in the case of loglinear Gaussian frameworks, the shock elasticity exactly corresponds to the impulse response function for Y_t.

Borovička and Hansen (2013) show that when the law of motion ψ for the state vector has the functional form generated by the series expansion method, closed form solutions for the shock elasticities exist, including the distribution of the shock elasticity function under the stationary distribution of the state vector. This holds both under the approximating model as well as under the worst case model given by (31). Evaluating the approximate solutions using the shock elasticity functions is therefore straightforward in our framework.

6 A quantitative example

Performance of approximation methods is typically shown on examples of simple general equilibrium models, like different versions of the textbook real business cycle model. But these simples models have virtually no effects beyond second order. We therefore compare the performance of the robust second-order approximation against a third-order approximation of the model by Bidder and Smith (2012). Bidder and Smith (2012) solve their model using a standard smooth third-order perturbation. We show that our second-order expansion generates quantity responses that are very close to those produced by Bidder and Smith (2012). We also analyze the pricing implications of this model and argue that the prices of risk in this model are two low to produce quantitatively relevant premia on risky assets.

6.1 The model

The model consists of a utility-maximizing household that exhibits a concern for robustness and an aggregate production technology with capital accumulation that is subject to convex adjustment costs. The technology is driven by a productivity shock with a permanent component and stochastic volatility. Given the structure of the economy, we can find the equilibrium allocation by solving an associated planner's problem.

The representative household has preferences over consumption C_t and labor L_t and its concern for robustness is expressed by a recursion for the continuation value (4), with the period utility function

$$u_t = \log\left(C_t - \xi C_{t-1} - \eta_0 L_t^{\eta_1} J_t\right)$$

where $J_t = C_t^{\nu} J_{t-1}^{1-\nu}$ is a Jaimovich and Rebelo (2009) adjustment factor that adjusts the disutility from labor to keep the preferences consistent with a balanced growth path. The technology by an exogenous labor augmenting productivity shock

$$\begin{aligned} z_{t+1} &= \Lambda_z + z_t + e^{v_{t+1}} \bar{\sigma}_z \varepsilon_{z,t+1} \\ v_{t+1} &= \rho v_t + \bar{\sigma}_v \varepsilon_{v,t+1} \end{aligned}$$

and produces output using the Cobb-Douglas production function

$$Y_t = \left(h_t K_t\right)^{\alpha} \left(e^{z_t} L_t\right)^{1-\alpha}$$

where h_t represents the utilization rate of capital in place. The stock of capital is accumulated according to

$$K_{t+1} = (1 - \delta(h_t)) K_t + I_t \left(1 - \frac{\kappa_I}{2}\left(\frac{I_t}{I_{t-1}} - e^{\Lambda_z}\right)^2\right)$$

where the second term on the right-hand side represents second-order adjustment costs and the function $\delta(h_t)$ is the utilization-dependent depreciation rate

$$\delta(h_t) = \delta_0 + \frac{\delta_1}{1+\delta_2} h_t^{1+\delta_2}.$$

Consumption and investment are restricted by the aggregate feasibility constraint

$$Y_t = C_t + I_t.$$

6.2 Stochastic discount factor and belief distortions

We focus closely on the dynamics of the stochastic discount factor of the representative household. The one-period stochastic discount factor consists of two components

$$S_{t+1} = \tilde{S}_{t+1} M_{t+1}$$

where $\tilde{S}_{t+1}$ is the discount ratio of marginal utilities of consumption

$$\tilde{S}_{t+1} = \beta \frac{\partial u_{t+1}/\partial C_{t+1}}{\partial u_t/\partial C_t}$$

and M_{t+1} is the belief distortion generated by the worst-case model and represented by the exponential tilt (3).

6.3 Comparison of results

Bidder and Smith (2012) focus on the quantity dynamics under the approximating and the worst-case model, where the former is associated with the data-generating process. Figure 2 plots nonlinear impulse responses from the Bidder and Smith (2012) paper in blue and compares them to our shock elasticities in red. While Bidder and Smith (2012) only plot the average impulse response, we plot both the average shock elasticities as well as their quantiles under the stationary distribution of the state vector.

The results show that the performance of the robust approximation effectively corresponds to that of a standard approximation with an order of approximation higher by one. This is particularly pronounced for the responses to the volatility shocks. Under a standard second-order approximation, the responses to volatility shocks would be zero by construc-

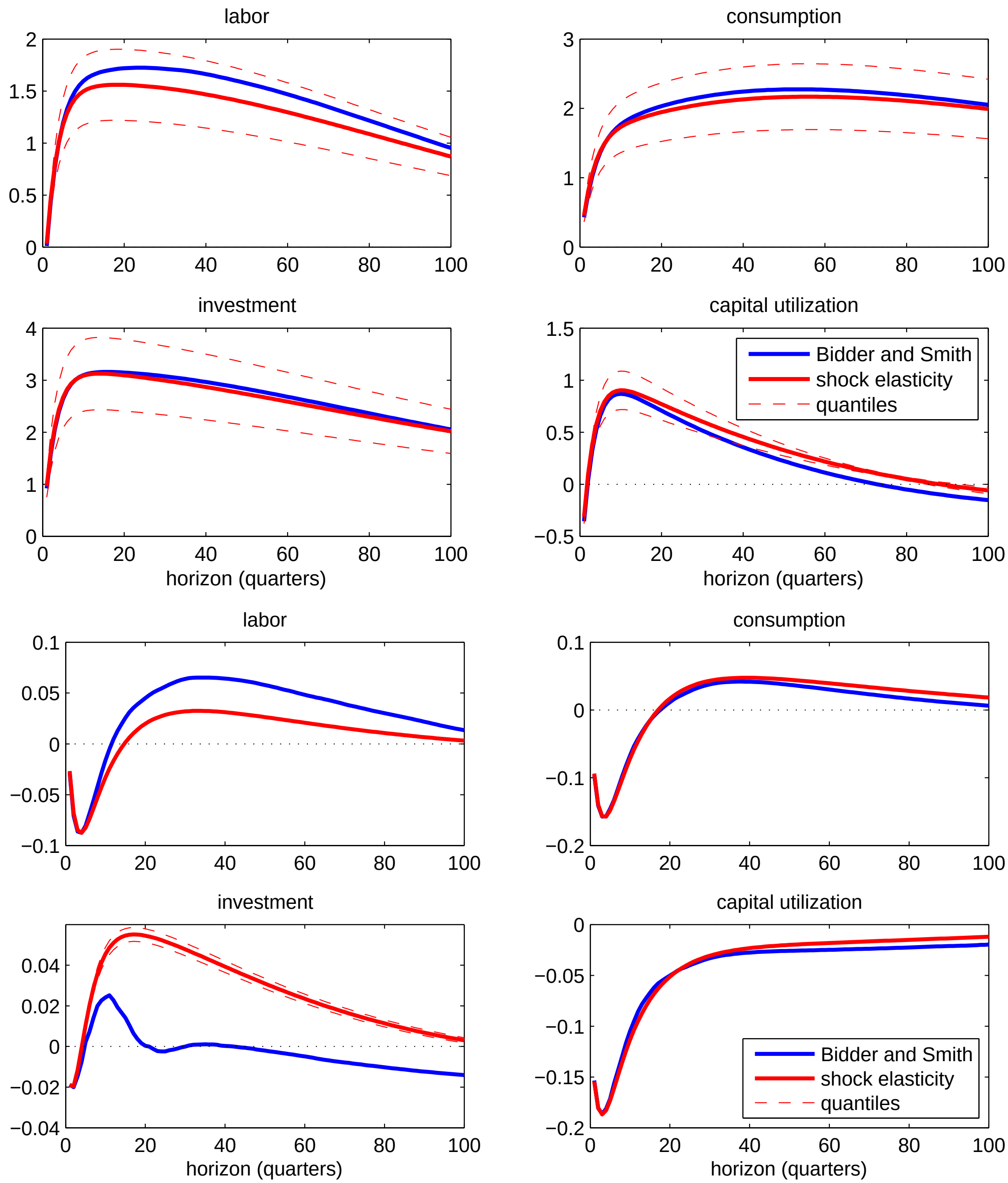

Figure 2: A comparison of shock elasticities computed using the second-order robust preference expansion method from this paper with the results from Bidder and Smith (2012). Top four graphs present responses to the technology shock, bottom four graphs responses to the volatility shock. Shock elasticities computed under the approximating model.

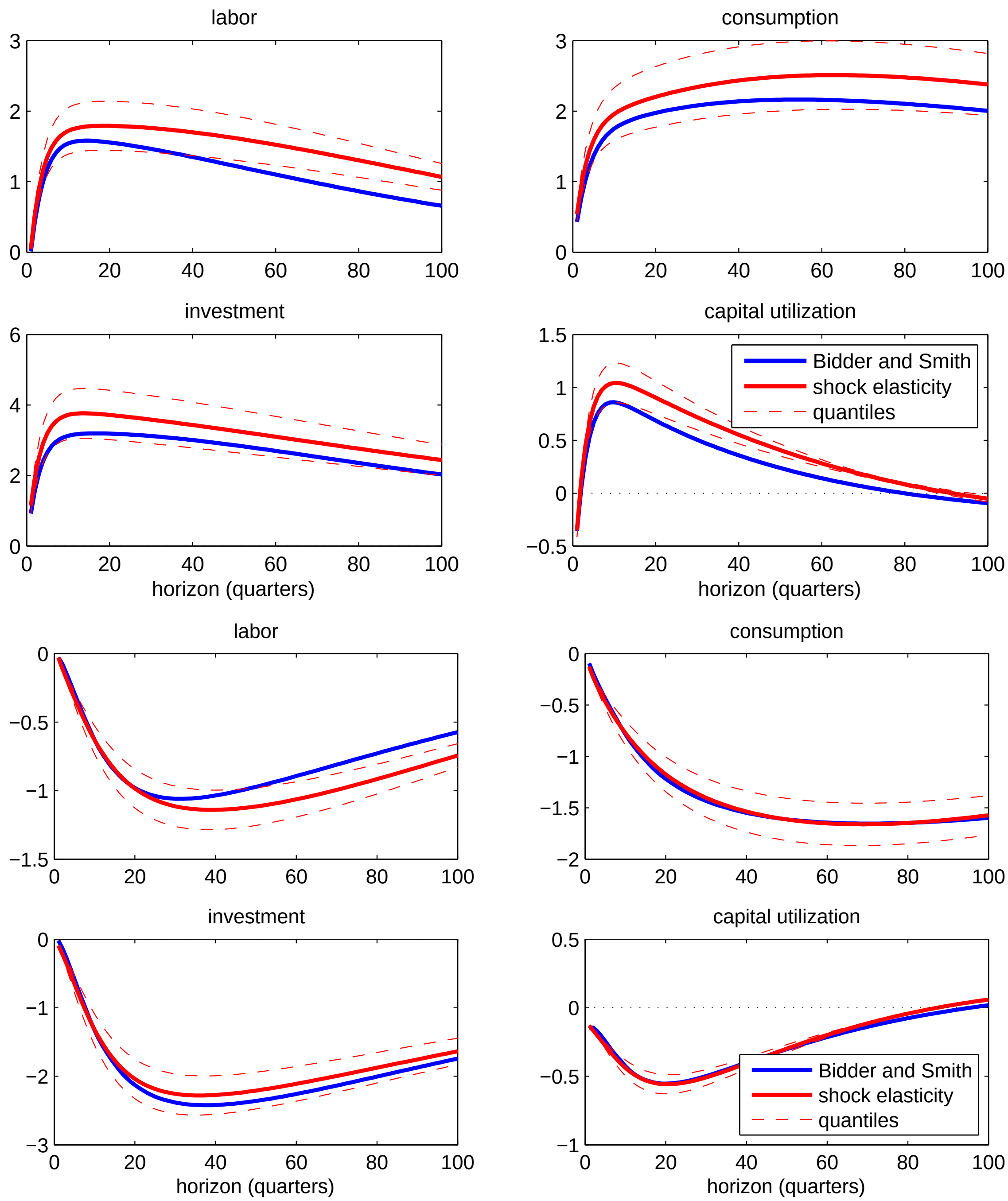

Figure 3: A comparison of shock elasticities computed using the second-order robust preference expansion method from this paper with the results from Bidder and Smith (2012). Top four graphs present responses to the technology shock, bottom four graphs responses to the volatility shock. Shock elasticities computed under the worst-case model, and the mean and quantiles of the responses are conditioned on the stationary distribution under the worst-case model.

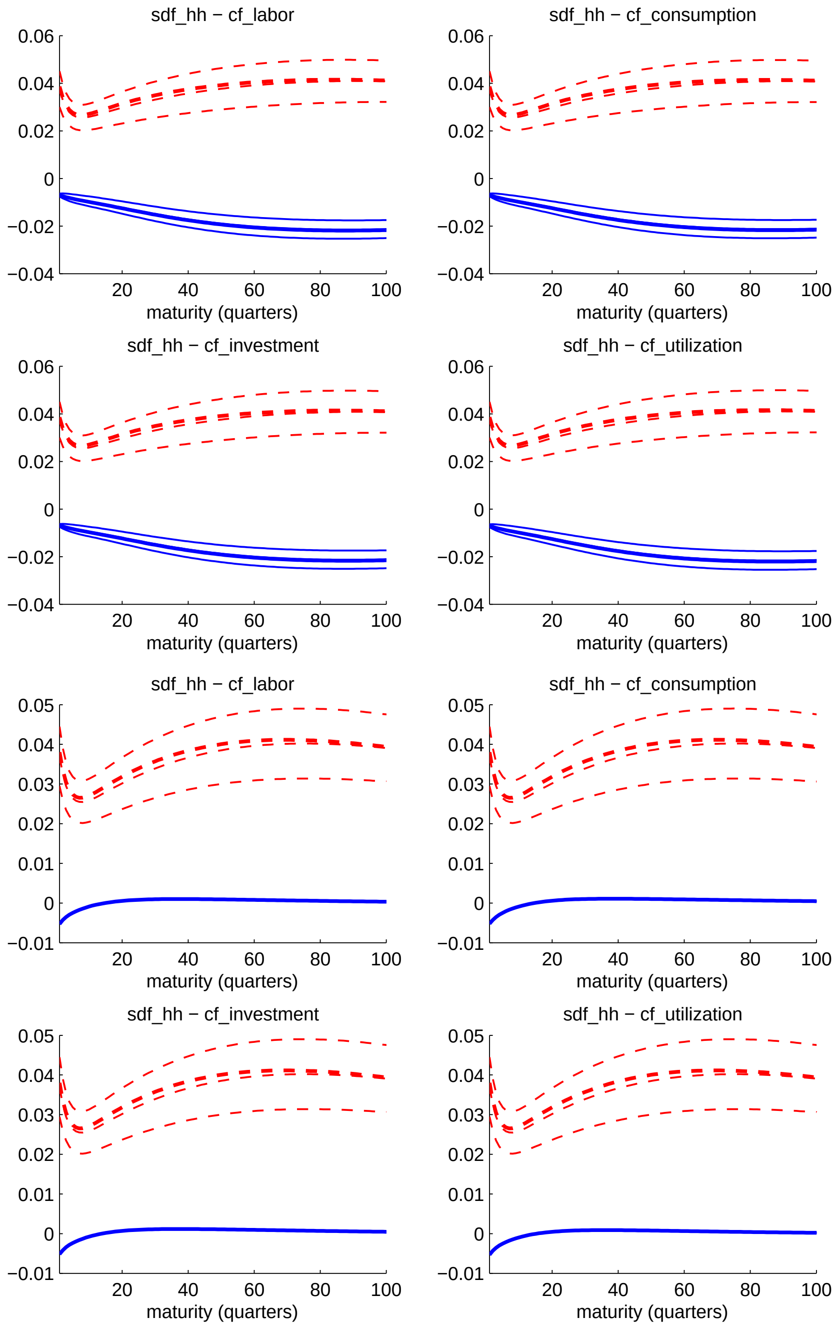

Figure 4: **Shock-price elasticities**: Shock-price elasticities for the technology shock (in red) and volatility shock (in blue). The upper two rows correspond to the robust agent, and are calculated by taking just the consumption part of the stochastic discount factor (without the distortion) and computing the shock-price elasticities under the worst-case model. The bottom two rows correspond to the shock-price elasticities for an agent who has the same equilibrium consumption process but his beliefs are not distorted toward the worst-case model. In effect, all shock-price elasticities here are computing using the same SDF, upper two rows under the worst-case model, bottom two rows under the approximating model.

tion. The robust preference expansion method pins down, in a second-order approximation, the volatility responses computed from a third order approximation.

Figure 3 shows that the approximation is good not only under the approximating model but also under the worst-case model of the robust agent.

With the belief distortion at hand, we can also conduct pricing exercises. For instance, the ratio of marginal utilities of consumption, $\tilde{S}$, corresponds to an equilibrium stochastic discount factor of a non-robust household who faces the same consumption and volatility processes as the robust household. The whole product, $S_{t+1} = \tilde{S}_{t+1} M_{t+1}$ is then the stochastic discount factor that the robust household is facing.

Figure 4 plots the shock-price elasticities (responses of expected returns to structural shocks). In line with our previous discussion, it is particularly the volatility shocks that the worst-case accentuates significantly — while for the non-robust agent, the volatility shock plays a negligible role in pricing (bottom four graphs), the shock carries a much higher price of risk (as measured by the shock-price elasticity) when robust concern is taken into account (top four graphs).

7 Concluding thoughts

In this paper, we showed that a judicious choice of the method for the approximation of the stochastic discount factor can decrease the order of approximation needed to obtain accurate solutions for relevant features of the model of interest. In particular, we consider models in which agents have a concern for robustness, reflected in the slanting of the subjective probability distribution toward worse outcomes. This probability distortion is a function of a parameter that controls the degree of robustness. We show how to derive a generalization of the series expansion method that scales this parameter jointly with the volatility of the shock. The method allows for multiple endogenously determined belief distortions, resulting in heterogeneous worst-case models of individual agnets. The resulting nonlinearities in the approximate equilibrium conditions can be handled analytically, using methods similar to those for perturbation solutions of rational expectations models.

We test the performance of the method by comparing nonlinear responses to economic shocks in an equilibrium model of Bidder and Smith (2012), who use a third-order perturbation approximation to capture the dynamic effects of shocks to stochastic volatility. We show that our method performs well in capturing responses to shocks both to the level as well as the volatility of the productivity process, using only a second-order approximation. The responses are close to those in Bidder and Smith (2012) both for the dynamics under the approximating as well as the worst-case model. The proximity of the volatility responses to those generated by the third-order approximation is particularly remarkable, as a conventional perturbation approximation method computed to the second order would lead to responses to a volatility shock that are exactly zero — there would be no stochastic volatility effects whatsoever.

In addition, our method has the advantage that it generates recursively linear second-order dynamics which we can use to compute the dynamic responses analytically. To do so, we use the concept of shock elasticities that we developed in our earlier work in Borovička and Hansen (2013). The shock elasticities measure the sensitivity of expected cash-flows (and expected returns) to the exposure of these cash-flows to economic shocks. Analytical tractability implies that in order to compute the responses, we do not need to rely on simulations of the equilibrium state vector and on techniques that discipline the unstable simulated trajectories.

While the method performs remarkably well for the analyzed example, it is important to understand the asymmetry in the approximation technique. It is based on the idea that the crucial nonlinearity of the model arises from the dynamics of the belief distortion in the stochastic discount factor, and therefore our method of approximation will significantly improve the solution of the whole model. For instance, nonlinearities arising from continuation-value components in recursive utility stochastic discount factors can be handled analogously but if the source of nonlinearity comes from other components of the model, the method may likely perform closer to the standard perturbation approximation.

The functional form of the solution that we generate can also be exploited advantageously in estimation techniques. Aruoba et al. (2012) consider such second-order recursively linear dynamics to develop tractable estimation techniques that explore nonlinearities in the solutions of DSGE models. Further work in this direction is likely to generate new promising insights.

Appendix

A Preliminary algebra

In this appendix, we provide some details on some algebraic operations used in the paper.algebraic rules of tensor algebra.

A.1 Definitions

To simplify work with Kronecker products, we define two operators vec and $\mathrm{mat}_{m,n}$. For an $m \times n$ matrix H, $\mathrm{vec}\,(H)$ produces a column vector of length mn created by stacking the columns of H:

$$h_{(j-1)m+i} = [\mathrm{vec(H)}]_{(j-1)m+i} = H_{ij}.$$

For a vector (column or row) h of length mn, $\mathrm{mat}_{m,n}\,(h)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$H_{ij} = [\mathrm{mat}_{m,n}(h)]_{ij} = h_{(j-1)m+i}.$$

We drop the m, n subindex if the dimensions of the resulting matrix are obvious from the context. For a square matrix A, define the sym operator as

$$\mathrm{sym}\,(A) = \frac{1}{2}\left(A + A'\right).$$

Apart from the standard operations with Kronecker products, notice that the following is true. For a row vector $H_{1\times nk}$ and column vectors $X_{n\times 1}$ and $W_{n\times 1}$

$$H\,(X \otimes W) = X'\,[\mathrm{mat}_{k,n}\,(H)]'\,W$$

and for a matrix $A_{n\times k}$, we have

$$X'AW = \left(\mathrm{vec}A'\right)'(X \otimes W)\,. \tag{32}$$

Also, for $A_{n\times n}$, $X_{n\times 1}$, $K_{k\times 1}$, we have

$$\begin{aligned} (AX) \otimes K &= (A \otimes K)\,X \\ K \otimes (AX) &= (K \otimes A)\,X. \end{aligned}$$

Finally, for column vectors $X_{n\times 1}$ and $W_{k\times 1}$,

$$(AX) \otimes (BW) = (A \otimes B)\,(X \otimes W)$$

and

$$(BW) \otimes (AX) = [B \otimes A_{\bullet j}]_{j=1}^{n} (X \otimes W)$$

where

$$[B \otimes A_{\bullet j}]_{j=1}^{n} = [B \otimes A_{\bullet 1} \quad B \otimes A_{\bullet 2} \quad \ldots \quad B \otimes A_{\bullet n}] .$$

A.2 Concise notation for derivatives

Consider a vector function $f(x, w)$ where x and w are column vectors of length m and n, respectively. The first-derivative matrix f_i where $i = x, w$ is constructed as follows. The k-th row $[f_i]_{k\bullet}$ corresponds to the derivative of the k-th component of f

$$[f_i(x, w)]_{k\bullet} = \frac{\partial f^{(k)}}{\partial i'}(x, w) .$$

Similarly, the second-derivative matrix is the matrix of vectorized and stacked Hessians of individual components with k-th row

$$[f_{ij}(x, w)]_{k\bullet} = \left(\text{vec} \frac{\partial^2 f^{(k)}}{\partial j \partial i'}(x, w) \right)' .$$

It follows from formula (32) that, for example,

$$x' \left(\frac{\partial^2 f^{(k)}}{\partial x \partial w'}(x, w) \right) w = \left(\text{vec} \frac{\partial^2 f^{(k)}}{\partial w \partial x'}(x, w) \right)' (x \otimes w) = [f_{xw}(x, w)]_{k\bullet} (x \otimes w) .$$

A.3 Conditional expectations

Let $w_{t+1} \sim N(0, I_k)$. Notice that a complete-the squares argument implies that, for a $1 \times k$ vector A, a $1 \times k^2$ vector B, and a scalar function $f(w)$,

$$\begin{aligned} & E_t [\exp(B(w_{t+1} \otimes w_{t+1}) + Aw_{t+1}) f(w_{t+1})] = \\ & \quad = E_t \left[\exp\left(\frac{1}{2} w_{t+1}' (\text{mat}_{k,k}(2B)) w_{t+1} + Aw_{t+1} \right) f(w_{t+1}) \right] = \\ & \quad = |I_k - \text{sym}[\text{mat}_{k,k}(2B)]|^{-1/2} \exp\left(\frac{1}{2} A (I_k - \text{sym}[\text{mat}_{k,k}(2B)])^{-1} A' \right) \cdot \tilde{E}_t [f(w_{t+1})] \end{aligned} \tag{33}$$

where under the $\tilde{}$ measure

$$w_{t+1} \sim N\left((I_k - \text{sym}[\text{mat}_{k,k}(2B)])^{-1} A', (I_k - \text{sym}[\text{mat}_{k,k}(2B)])^{-1} \right) . \tag{34}$$

B Series expansions

The series expansion method analyzes a small-noise approximation of the model $x_t = \psi(x_{t-1}, w_t)$ where $w_t \sim N(0, I_k)$ and x_t is an $n \times 1$ Markov state around a deterministic path. We consider a class of models

$$x_t(\mathsf{q}) = \psi(x_{t-1}(\mathsf{q}), \mathsf{q} w_t, \mathsf{q})$$

as a function of the perturbation parameter q. Repeated differentiation with respect to q and evaluation of the derivatives at $\mathsf{q} = 0$ yields a sequence of dynamic models (here, up to the second order)

$$\begin{aligned} x_{0t} &= \psi(x_{0t-1}, 0) \\ x_{1t} &= \psi_x x_{1t-1} + \psi_w w_t + \psi_q \\ x_{2t} &= \psi_x x_{2t-1} + \psi_{xx}(x_{1t-1} \otimes x_{1t-1}) + 2\psi_{xw}(x_{1t-1} \otimes w_t) + 2\psi_{xq} x_{1t-1} + \\ &\quad + \psi_{ww}(w_t \otimes w_t) + 2\psi_{wq} w_t + \psi_{qq} \end{aligned} \tag{35}$$

where

$$x_t(\mathsf{q}) \approx x_{0t} + \mathsf{q} x_{1t} + \frac{\mathsf{q}^2}{2} x_{2t}$$

and x_{jt} can thus be interpreted as derivatives of the proces x_t with respect to q. The matrices ψ_i and ψ_{ij} are first- and second-order derivatives of ψ, constructed using the notation from Appendix A.2. Observe that the approximate dynamics up to j-th order is a Markov system with a $(j+1)n$ dimensional state vector. $x_t^j = (x_{0t}', x_{1t}', \ldots, x_{jt})'$.

Functions of the state vector $f(x_t)$ are approximated in the same way. For instance, agent's i period utility function $u_t^i = u^i(x_t)$ yields the approximation

$$\begin{aligned} u_{0t}^i &= u^i(x_{0t}) \\ u_{1t}^i &= u_x^i x_{1t} + u_q^i \\ u_{2t}^i &= u_x^i x_{2t} + u_{xx}^i(x_{1t} \otimes x_{1t}) + 2u_{xq}^i x_{1t} + u_{qq}^i \end{aligned}$$

Here, we preserve the possibility of an explicit dependence of u^i on q. Although in applications the partial derivatives of u^i with respect to q will almost always be zero, this will no longer be true for endogenously determined quantities in our framework. Also note that the j-th order approximation approximation preserves the Markov property with state vector x_t^j.

C Approximate solution

In this section of the appendix, we provide details on the solution of the model (12). We start with the construction of the continuation values and worst-case belief distortions, and then discuss the solution for the equilibrium conditions.

C.1 Continuation values and worst-case distortions

In Section 2.1, we derived recursion (4) for the continuation value of an agent with a concern for robustness, which suggests the expansion

$$V_t^i(\mathsf{q}) = u_t^i(\mathsf{q}) - \beta_i\theta_i \log E_t\left[\exp\left(-\frac{1}{\theta_i}V_{t+1}^i(\mathsf{q})\right)\right].$$

Repeated differentiation of

$$V_t^i(\mathsf{q}) \approx V_{0t}^i + \mathsf{q}V_{1t}^i + \frac{\mathsf{q}^2}{2}V_{2t}^i$$

leads to the set of recursive formulas

$$\begin{aligned}
V_{0t}^i &= u_{0t}^i + \beta V_{0t+1}^i \\
V_{1t}^i &= u_{1t}^i - \beta_i\theta_i \log E_t\left[\exp\left(-\frac{1}{\theta_i}V_{1t+1}^i\right)\right] && (36)\\
V_{2t}^i &= u_{2t}^i + \beta_i E_t\left[\frac{\exp\left(-\frac{1}{\theta_i}V_{1t+1}^i\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}V_{1t+1}^i\right)\right]}V_{2t+1}^i\right]. && (37)
\end{aligned}$$

Since $u_t^i = u^i(x_t)$, we can make the guess that $V_t^i(\mathsf{q}) = V^i(x_t(\mathsf{q}),\mathsf{q})$ which leads to the following expressions for the derivatives of V_t^i:

$$\begin{aligned}
V_{1t}^i &= V_x^i x_{1t} + V_q^i \\
V_{2t}^i &= V_x^i x_{2t} + V_{xx}^i(x_{1t}\otimes x_{1t}) + 2V_{xq}^i x_{1t} + V_{qq}^i
\end{aligned}$$

where we ignored the expression for the deterministic term V_{0t}^i as it is irrelevant for our calculations.

We plug these expressions into the recursion (36), substitute for x_{1t+1} from the law of motion (35), solve for the log normal formulas and compare coefficients on x_{1t} and the constant term to obtain solutions for V_x^i and V_q^i

$$\begin{aligned}
V_x^i &= u_x^i(I_n - \beta_i\psi_x)^{-1} && (38)\\
V_q^i &= (1-\beta_i)^{-1}\left[u_q^i + \beta_i V_x^i\psi_q - \beta_i\frac{1}{2\theta_i}V_x^i\psi_w\psi_w'\left(V_x^i\right)'\right].
\end{aligned}$$

With this solution, we can also solve for the order zero distortion for the worst-case model of agent i in expression (16).

$$M_{0t+1}^i = \frac{\exp\left(-\frac{1}{\theta_i}V_{1t+1}^i\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}V_{1t+1}^i\right)\right]} = \frac{\exp\left(-\frac{1}{\theta_i}V_x^i\psi_w w_{t+1}\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}V_x^i\psi_w w_{t+1}\right)\right]} \quad (39)$$

where ψ_w will be determined later. This change of measure that imposes a new drift on w_{t+1} under the worst-case model, so that $w_{t+1} \sim N\left(-\frac{1}{\theta_i}\left(V_x^i \psi_w\right)', I_k\right)$, i.e.,

$$\begin{aligned}
\tilde{E}_t^i\left[w_{t+1}\right] &= -\frac{1}{\theta_i}\left(V_x^i \psi_w\right)' \qquad (40)\\
\tilde{E}_t^i\left[w_{t+1} \otimes w_{t+1}\right] &= \tilde{E}_t^i\left[vec\left(w_{t+1} w_{t+1}'\right)\right] = vec\left(\widetilde{Var}_t^i\left(w_{t+1}\right) + \tilde{E}_t^i\left[w_{t+1}\right] \tilde{E}_t^i\left[w_{t+1}'\right]\right) = \\
&= vec\left(I_k + \frac{1}{\left(\theta_i\right)^2}\left(V_x^i \psi_w\right)'\left(V_x^i \psi_w\right)\right)
\end{aligned}$$

Equation (37) can therefore be written as

$$V_{2t}^i = u_{2t}^i + \beta_i \tilde{E}_t^i\left[V_{2t+1}^i\right] \qquad (41)$$

where $\tilde{E}_t^i[\cdot]$ is the expectation induced by the distortion M_{0t+1}^i.

Substituting the expansions into equation (41) yields

$$\begin{aligned}
V_x^i x_{2t} + V_{xx}^i\left(x_{1t} \otimes x_{1t}\right) + 2 V_{xq}^i x_{1t} + V_{qq}^i = u_x^i x_{2t} + u_{xx}^i\left(x_{1t} \otimes x_{1t}\right) + 2 u_{xq}^i x_{1t} + u_{qq}^i + \\
+\beta_i \tilde{E}_t^i\left[V_x^i x_{2t+1} + V_{xx}^i\left(x_{1t+1} \otimes x_{1t+1}\right) + 2 V_{xq}^i x_{1t+1} + V_{qq}^i\right]
\end{aligned}$$

The individual terms in the expectations are equal to

$$\begin{aligned}
&\tilde{E}_t^i\left[V_x^i x_{2t+1}\right] = V_x^i \psi_x x_{2t} + V_x^i \psi_{xx}\left(x_{1t} \otimes x_{1t}\right) + \left(2 V_x^i \psi_{xq} + \tilde{E}_t^i\left[w_{t+1}'\right]\left(\operatorname{mat}_{k,n}\left(2 V_x^i \psi_{xw}\right)\right)\right) x_{1t} + \\
&\qquad +V_x^i \psi_{qq} + V_x^i \psi_{ww} \tilde{E}_t^i\left[w_{t+1} \otimes w_{t+1}\right] + 2 V_x^i \psi_{wq} \tilde{E}_t^i\left[w_{t+1}\right] \\
&\tilde{E}_t^i\left[V_{xx}^i\left(x_{1t+1} \otimes x_{1t+1}\right)\right] = V_{xx}^i\left(\psi_x \otimes \psi_x\right)\left(x_{1t} \otimes x_{1t}\right) + V_{xx}^i\left(\psi_x \otimes \psi_q + \psi_q \otimes \psi_x\right) x_{1t} + \\
&\qquad +\tilde{E}_t^i\left[w_{t+1}'\right]\left[\operatorname{mat}_{k,n}\left(V_{xx}^i\left(\psi_x \otimes \psi_w\right)\right) + \left(\operatorname{mat}_{n,k}\left(V_{xx}^i\left(\psi_w \otimes \psi_x\right)\right)\right)'\right] x_{1t} + \\
&\qquad +V_{xx}^i\left[\left(\psi_q \otimes \psi_q\right) + \left(\psi_w \otimes \psi_w\right) \tilde{E}_t^i\left[w_{t+1} \otimes w_{t+1}\right] + \left(\psi_w \otimes \psi_q + \psi_q \otimes \psi_w\right) \tilde{E}_t^i\left[w_{t+1}\right]\right] \\
&\tilde{E}_t^i\left[2 V_{xq}^i x_{1t+1}\right] = 2 V_{xq}^i \psi_x x_{1t} + 2 V_{xq}^i\left(\psi_w \tilde{E}_t^i\left[w_{t+1}\right] + \psi_q\right) \\
&\tilde{E}_t^i\left[V_{qq}^i\right] = V_{qq}^i
\end{aligned}$$

Now we can compare coefficients on x_{2t}, $x_{1t} \otimes x_{1t}$, x_{1t} and the constant term in V_{2t}^i. The coefficient on x_{2t} yields the same equation as (38), and the remaining coefficients yield the following equations for V^i_{xx}, V^i_{xq} and V^i_{qq}:

$$
\begin{aligned}
& V^i_{xx} = \left[u^i_{xx} + \beta_i V^i_x \psi_{xx}\right]\left[I_{n^2} - \beta_i\left(\psi_x \otimes \psi_x\right)\right]^{-1} \qquad (42)\\
& V^i_{xq}\left[I_n - \beta_i \psi_x\right] = u^i_{xq} + \beta_i\left(V^i_x \psi_{xq} + \frac{1}{2}V^i_{xx}\left(\psi_x \otimes \psi_q + \psi_q \otimes \psi_x\right)\right) + \\
& \qquad -\beta_i \frac{1}{2\theta_i}\left(V^i_x \psi_w\right)\left[\mathrm{mat}_{k,n}\left(2V^i_x \psi_{xw} + V^i_{xx}\left(\psi_x \otimes \psi_w\right)\right) + \left(\mathrm{mat}_{n,k}\left(V^i_{xx}\left(\psi_w \otimes \psi_x\right)\right)\right)'\right]\\
& V^i_{qq}\left(1 - \beta_i\right) = u^i_{qq} + \beta_i\left(V^i_x \psi_{qq} + V^i_{xx}\left(\psi_q \otimes \psi_q\right) + 2V^i_{xq}\psi_q\right) + \\
& \qquad -\beta_i \frac{1}{\theta_i}\left(2V^i_x \psi_{wq} + V^i_{xx}\left(\psi_w \otimes \psi_q + \psi_q \otimes \psi_w\right) + 2V^i_{xq}\psi_w\right)\left(V^i_x \psi_w\right)' + \\
& \qquad +\beta_i\left(V^i_x \psi_{ww} + V^i_{xx}\left(\psi_w \otimes \psi_w\right)\right) vec\left(I_k + \frac{1}{\left(\theta_i\right)^2}\left(V^i_x \psi_w\right)'\left(V^i_x \psi_w\right)\right)
\end{aligned}
$$

These expressions depend on the unknown derivatives of ψ but we show below how to proceed in order to in order to construct these terms sequentially.

C.2 Approximation of equilibrium conditions

We now approximate the system of equilibrium conditions

$$
E_t\left[\mathbb{M}_{t+1} g\left(x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t\right)\right] \qquad (43)
$$

where $\mathbb{M}_{t+1} = \left\{M^{\sigma_1}_{t+1}, \ldots M^{\sigma_n}_{t+1}\right\}$ is a diagonal matrix of belief distortions and $\sigma_j \in \{0, 1, \ldots, I\}$ index belief distortions of agents $i \in \{1, \ldots, I\}$ with $M^0_{t+1} \equiv 1$. These belief distortions are given by expression (39).

It will be useful to define

$$
\begin{aligned}
\tilde{\mathbb{E}}_t\left[w_{t+1}\right] &= \left[\tilde{E}^{\sigma_1}_t\left[w_{t+1}\right] \ \ldots \tilde{E}^{\sigma_n}_t\left[w_{t+1}\right]\right] = \\
&= \left[E_t\left[M^{\sigma_1}_{0t+1} w_{t+1}\right] \ \ldots \ E_t\left[M^{\sigma_1}_{0t+1} w_{t+1}\right] \ \right]
\end{aligned}
$$

the matrix consisting of columns that represent conditional means of w_{t+1} under distortions associated with individual equations of the set of equilibrium conditions $0 = E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right]$. These individual columns are constructed in equation (40). We also define accordingly the matrix of second-moment vectors

$$
\tilde{\mathbb{E}}_t\left[w_{t+1} \otimes w_{t+1}\right] = \left[\tilde{E}^{\sigma_1}_t\left[w_{t+1} \otimes w_{t+1}\right] \ \ldots \tilde{E}^{\sigma_n}_t\left[w_{t+1} \otimes w_{t+1}\right]\right].
$$

Expanding the set of equilibrium conditions (43) yields

$$
\begin{aligned}
0 &= E_t\left[\mathbb{M}_{0t+1} g_{0t+1}\right] = g_{0t+1}\\
0 &= E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right] + E_t\left[\mathbb{M}_{1t+1} g_{0t+1}\right] = E_t\left[\mathbb{M}_{0t+1} g_{1t+1}\right]\\
0 &= E_t\left[\mathbb{M}_{0t+1} g_{2t+1}\right] + 2E_t\left[\mathbb{M}_{1t+1} g_{1t+1}\right] + E_t\left[\mathbb{M}_{2t+1} g_{0t+1}\right] = E_t\left[\mathbb{M}_{0t+1} g_{2t+1}\right] + 2E_t\left[\mathbb{M}_{1t+1} g_{1t+1}\right]
\end{aligned}
$$

and the second equalities in each line use the facts that g_{0t+1} is deterministic, $E_t[\mathbb{M}_{0t+1}] = I_n$ and $E_t[\mathbb{M}_{1t+1}] = E_t[\mathbb{M}_{1t+1}] = 0$. The first equation is a system of nonlinear equations for the deterministic path around which we approximate the model, typically a constant steady state.

We will deal with the remaining pair of equations. Notice that the diagonal elements of $\mathbb{M}_{0t+1}$ are given (with the appropriate reindexing by σ_j) by (39) and depend on the unknown derivative ψ_w, while the elements of $\mathbb{M}_{1t+1}$ represented in (16) depend on the second-order derivatives of ψ.

C.2.1 First-order equations

For the first-order derivative of the equilibrium conditions, we have

$$0 = E_t[\mathbb{M}_{0t+1} g_{1t+1}] \tag{44}$$

The first-order expansion of g_1 is

$$\begin{aligned} g_{1t+1} &= g_{x+}x_{1t+1} + g_x x_{1t} + g_{x-}x_{1t-1} + g_{w+}w_{t+1} + g_w w_t + g_q = \\ &= [(g_{x+}\psi_x + g_x)\psi_x + g_{x-}]\,x_{1t-1} + [(g_{x+}\psi_x + g_x)\psi_w + g_w]\,w_t + \\ &\quad + (g_{x+}\psi_x + g_{x+} + g_x)\psi_q + g_q + (g_{x+}\psi_w + g_{w+})\,w_{t+1} \end{aligned}$$

where symbols x_+, x, x_-, w_+, w, q represent partial derivatives with respect to $x_{t+1}, x_t, x_{t-1}, w_{t+1}, w_t$ and q, respectively Equation (44) thus is a system of linear second-order stochastic difference equations. There are well-known results that discuss the conditions under which there exists a unique stable solution to this system. We assume that such conditions are satisfied. Comparing coefficients on x_{1t-1}, w_t and the constant term implies the set of equations

$$\begin{aligned} 0 &= (g_{x+}\psi_x + g_x)\psi_x + g_{x-} \\ 0 &= (g_{x+}\psi_x + g_x)\psi_w + g_w \\ 0 &= (g_{x+}\psi_x + g_{x+} + g_x)^i\psi_q + (g_q)^i + (g_{x+}\psi_w + g_{w+})^i\,\tilde{E}_t^i[w_{t+1}] \quad i = 1,\ldots,n \end{aligned} \tag{45}$$

where the i superindices denote i-th equation. The first equation is a quadratic matrix equation who can be solved for the stable solution using standard methods like the qz-decomposition (Sims (2002)). The second equation then implies a solution fow ψ_w

$$\psi_w = -(g_{x+}\psi_x + g_x)^{-1} g_w.$$

The important observation is the fact that equations for ψ_x and ψ_w do not depend on the belief distortion matrix $\tilde{\mathbb{E}}_t[w_{t+1}]$, and thus their solution do not differ from a rational expectations solution. We can therefore solve for ψ_w, then construct the belief distortion matrix $\mathbb{M}_{0t+1}$ using (39) and finally compute the constant term

$$\psi_q = -\left(g_{x+}\psi_x + g_{x+} + g_x\right)^{-1}\left[g_q + \operatorname{diag}\left(\left(g_{x+}\psi_w + g_{w+}\right)\tilde{\mathbb{E}}_t\left[w_{t+1}\right]\right)\right]$$

where the diag $(\cdot)$ operator generates a column vector from the diagonal of a matrix.

C.2.2 Second-order equations

Now we use equation

$$0 = E_t\left[\mathbb{M}_{0t+1}g_{2t+1}\right] + 2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right] \tag{46}$$

to solve for the second-order dynamics given by x_{2t-1}. The algorithm is analogous to that for the first-order dynamics. Plugging in expansions of g_{1t+1} and g_{2t+1}, and substituting in the laws of motion (35) yields a set of equations in x_{1t-1}, x_{2t-1} and w_t. Then we can compare coefficients on terms x_{1t-1}, $x_{1t-1}\otimes x_{1t-1}$, $x_{1t-1}\otimes w_t$, w_t, $w_t\otimes w_t$ and the constant term to obtain solutions for ψ_{xx}, ψ_{xw}, ψ_{xq}, ψ_{ww}, ψ_{wq} and ψ_{qq}. We deal with the two terms in (46) separately.

First term First compute $E_t\left[\mathbb{M}_{0t+1}g_{2t+1}\right]$. The second-order derivative of g_{t+1} can be written as

$$g_{2t+1} = g_{x+}x_{2t+1} + g_x x_{2t} + g_{x-}x_{2t-1} + \sum_{i,j} g_{z_i z_j}\left(z_i \otimes z_j\right)$$

where $z_i, z_j \in \{x_{1t+1}, x_{1t}, x_{1t-1}, w_{t+1}, q\}$ and $z_i = q$ in the Kronecker product represents 1. Symmetry also implies that $g_{z_i z_j}\left(z_i \otimes z_j\right) = g_{z_j z_i}\left(z_j \otimes z_i\right)$. We need to compute the expectation of g_{2t+1}, taking into account the distorted expectation of w_{t+1} under $\mathbb{M}_{0t+1}$, and substitute repeatedly to obtain the expression as a function of x_{1t-1}, x_{2t-1} and w_t. As in the previous writeup, consider coefficients on individual terms. The following expansions of Kronecker products are useful:

$$\begin{aligned}
x_{1t+1}\otimes x_{1t+1} &= \left(\psi_x \otimes I_n\right)\left(x_{1t}\otimes x_{1t+1}\right) + \left[\psi_w \otimes \left(I_n\right)_{\cdot j}\right]_{j=1}^{n}\left(x_{1t+1}\otimes w_{t+1}\right) + \left(\psi_q \otimes I_n\right)x_{1t+1} \\
x_{1t}\otimes x_{1t+1} &= \left(I_n \otimes \psi_x\right)\left(x_{1t}\otimes x_{1t}\right) + \left(I_n \otimes \psi_w\right)\left(x_{1t}\otimes w_{t+1}\right) + \left(I_n \otimes \psi_q\right)x_{1t} \\
x_{1t+1}\otimes w_{t+1} &= \left(\psi_x \otimes I_k\right)\left(x_{1t}\otimes w_{t+1}\right) + \left(\psi_w \otimes I_k\right)\left(w_{t+1}\otimes w_{t+1}\right) + \left(\psi_q \otimes I_k\right)w_{t+1}
\end{aligned}$$

In order to simplify notation for the intermediate steps, denote Γ_{x2+}, Γ_{x2} and Γ_{x2-} the coefficients on x_{2t+1}, x_{2t} and x_{2t-1}, and $\Gamma_{z_i z_j}$ the coefficients on $z_i \otimes z_j$ with notation as above. These coefficients include terms that are the result of substitution for terms defined earlier.We denote in red the unknown coefficients that we need to solve for.

Coefficients on x_{t+1} terms:

$$\begin{aligned}
[x_{2t+1}] &: \Gamma_{x2+} = g_{x+} \\
[x_{1t+1} \otimes x_{1t+1}] &: \Gamma_{x+x+} = g_{x+x+} \\
[x_{1t+1} \otimes w_{t+1}] &: \Gamma_{x+w+} = 2g_{x+w+} + \Gamma_{x+x+} \left[\psi_w \otimes (I_n)_{.j}\right]_{j=1}^{n} \\
[x_{1t+1} \otimes w_t] &: \Gamma_{x+w} = 2g_{x+w} \\
[x_{1t+1}] &: \Gamma_{x+q} = 2g_{x+q} + \Gamma_{x+x+} (\psi_q \otimes I_n)
\end{aligned}$$

Coefficients on x_t terms:

$$\begin{aligned}
[x_{2t}] &: \Gamma_{x2} = g_x + \Gamma_{x2+}\psi_x \\
[x_{1t} \otimes x_{1t+1}] &: \Gamma_{xx+} = 2g_{xx+} + \Gamma_{x+x+} (\psi_x \otimes I_n) \\
[x_{1t} \otimes x_{1t}] &: \Gamma_{xx} = g_{xx} + \Gamma_{x2+}\psi_{xx} + \Gamma_{xx+} (I_n \otimes \psi_x) \\
[x_{1t} \otimes w_{t+1}] &: \Gamma_{xw+} = 2g_{xw+} + 2\Gamma_{x2+}\psi_{xw} + \Gamma_{x+w+} (\psi_x \otimes I_k) + \Gamma_{xx+} (I_n \otimes \psi_w) \\
[x_{1t} \otimes w_t] &: \Gamma_{xw} = 2g_{xw} + \Gamma_{x+w} (\psi_x \otimes I_k) + \Gamma_{xx} \left[\psi_w \otimes (I_n)_{.j}\right]_{j=1}^{n} \\
[x_{1t}] &: \Gamma_{xq} = 2g_{xq} + 2\Gamma_{x2+}\psi_{xq} + \Gamma_{x+q}\psi_x + \Gamma_{xx+} (I_n \otimes \psi_q) + \Gamma_{xx} (\psi_q \otimes I_n)
\end{aligned}$$

Coefficients on x_{t-1} terms:

$$\begin{aligned}
[x_{2t-1}] &: \Gamma_{x2-} = g_{x-} + \Gamma_{x2}\psi_x \\
[x_{1t-1} \otimes x_{1t+1}] &: \Gamma_{x-x+} = 2g_{x-x+} \\
[x_{1t-1} \otimes x_{1t}] &: \Gamma_{x-x} = 2g_{x-x} + \Gamma_{xx} (\psi_x \otimes I_n) + \Gamma_{x-x+} (I_n \otimes \psi_x) \\
[x_{1t-1} \otimes x_{1t-1}] &: \Gamma_{x-x-} = g_{x-x-} + \Gamma_{x2}\psi_{xx} + \Gamma_{x-x} (I_n \otimes \psi_x) \\
[x_{1t-1} \otimes w_{t+1}] &: \Gamma_{x-w+} = 2g_{x-w+} + \Gamma_{xw+} (\psi_x \otimes I_k) + \Gamma_{x-x+} (I_n \otimes \psi_w) \\
[x_{1t-1} \otimes w_t] &: \Gamma_{x-w} = 2g_{x-w} + 2\Gamma_{x2}\psi_{xw} + \Gamma_{xw} (\psi_x \otimes I_k) + \Gamma_{x-x} (I_n \otimes \psi_w) \\
[x_{1t-1}] &: \Gamma_{x-q} = 2g_{x-q} + 2\Gamma_{x2}\psi_{xq} + \Gamma_{xq}\psi_x + \Gamma_{x-x+} (I_n \otimes \psi_q) + \Gamma_{x-x} (I_n \otimes \psi_q)
\end{aligned}$$

Coefficients on terms:

$$\begin{aligned}
[w_{t+1} \otimes w_{t+1}] &: \Gamma_{w+w+} = g_{w+w+} + \Gamma_{x2+}\psi_{ww} + \Gamma_{x+w+} (\psi_w \otimes I_k) \\
[w_t \otimes w_{t+1}] &: \Gamma_{ww+} = 2g_{ww+} + \Gamma_{xw+} (\psi_w \otimes I_k) \\
[w_{t+1}] &: \Gamma_{w+q} = 2g_{w+q} + 2\Gamma_{x2+}\psi_{wq} + \Gamma_{x+w+} (\psi_q \otimes I_k) + \Gamma_{x+q}\psi_w + \Gamma_{xw+} (\psi_q \otimes I_k) \\
[w_t \otimes w_t] &: \Gamma_{ww} = g_{ww} + \Gamma_{x2}\psi_{ww} + \Gamma_{xw} (\psi_w \otimes I_k) \\
[w_t] &: \Gamma_{wq} = 2g_{wq} + \Gamma_{x+w} (\psi_q \otimes I_k) + 2\Gamma_{x2}\psi_{wq} + \Gamma_{xw} (\psi_q \otimes I_k) + \Gamma_{xq}\psi_w \\
[const] &: \Gamma_{qq} = g_{qq} + \Gamma_{x2+}\psi_{qq} + \Gamma_{x+q}\psi_q + \Gamma_{x2}\psi_{qq} + \Gamma_{xq}\psi_q \\
[w_{t+1} \otimes w_t] &: \Gamma_{w+w} = \Gamma_{x+w} (\psi_w \otimes I_k)
\end{aligned}$$

Collecting terms on individual coefficients will lead to linear equations for ψ_{ij} of the following type:

$$A\psi_{ij} + B\psi_{ij}C + D = 0$$

where the dimensionality is as follows:

$$\begin{array}{rcllll}
[\psi_{xx}]_{n\times n^2} & : & A_{n\times n} & B_{n\times n} & C_{n^2\times n^2} & D_{n\times n^2} \\
[\psi_{xw}]_{n\times nk} & : & A_{n\times n} & B_{n\times n} & C_{nk\times nk} & D_{n\times nk} \\
[\psi_{xq}]_{n\times n} & : & A_{n\times n} & B_{n\times n} & C_{n\times n} & D_{n\times n} \\
[\psi_{ww}]_{n\times k^2} & : & A_{n\times n} & B_{n\times n} & C_{k^2\times k^2} & D_{n\times k^2} \\
[\psi_{wq}]_{n\times k} & : & A_{n\times n} & B_{n\times n} & C_{k\times k} & D_{n\times k}
\end{array}$$

The solution to this equation is

$$vec\left(\psi_{ij}\right) = -\left[I \otimes A + C' \otimes B\right]^{-1} vec\left(D\right)$$

where I is an identity matrix of the same size as C.

The coefficient on $[x_{1t-1} \otimes x_{1t-1}]$ determines ψ_{xx}. After substituting the $\Gamma.$ terms, we obtain the equation

$$\begin{aligned} 0 \;=\;& \left(g_x + g_{x+}\psi_x\right)\psi_{xx} + g_{x+}\psi_{xx}\left(\psi_x \otimes \psi_x\right) + g_{x-x-} + 2g_{x-x}\left(I_n \otimes \psi_x\right) + \\ & + \left[g_{xx} + 2g_{xx+}\left(I_n \otimes \psi_x\right) + g_{x+x+}\left(\psi_x \otimes \psi_x\right)\right]\left(\psi_x \otimes \psi_x\right) + 2g_{x-x+}\left(I_n \otimes \left(\psi_x\right)^2\right) \end{aligned} \tag{47}$$

The coefficient on $[x_{1t-1} \otimes w_t]$ determines ψ_{xw}:

$$0 = 2\Gamma_{x2}\psi_{xw} + 2g_{x-w} + \Gamma_{xw}\left(\psi_x \otimes I_k\right) + \Gamma_{x-x}\left(I_n \otimes \psi_w\right) \tag{48}$$

where we do not need to expand the $\Gamma.$ terms further, as they do not depend on ψ_{xw}.

The coefficient on $[w_t \otimes w_t]$ determines the term ψ_{ww}:

$$0 = \Gamma_{x2}\psi_{ww} + g_{ww} + \Gamma_{xw}\left(\psi_w \otimes I_k\right) \tag{49}$$

What remains to be determined are coefficients ψ_{xq}, ψ_{wq} and ψ_{ww}. These are more complicated, as they are determined from equations for the coefficients on x_{1t-1}, w_t and the constant term, which also depend on the second term in (46). We discuss this second term $2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]$ later, however we note already now that the second term also depends on the second-order derivatives of the continuation values. For now, we only compute the contribution of the first term, $E_t\left[\mathbb{M}_{0t+1}g_{2t+1}\right]$, and reflect the fact that the contribution of the second term is missing by the symbol $\approx$ in the equations below.

Coefficients on $[x_{1t-1}]$ and $[x_{1t-1} \otimes w_{t+1}]$ determine the *contribution* of the first term in (46) to determining ψ_{xq}. The i-th equation in terms containing x_{1t-1} and $x_{1t-1} \otimes w_{t+1}$ reads

$$\Gamma^i_{x-q} x_{1t-1} + \tilde{E}^i_t \left[\Gamma^i_{x-w+} \left(x_{1t-1} \otimes w_{t+1}\right)\right]$$

Reorganizing the second term and stacking equation rows, we get

$$0 \approx \Gamma_{x-q} + \left[E^i_t \left[w'_{t+1}\right] \operatorname{mat}_{k,n} \left(\Gamma^i_{x-w+}\right)\right]^n_{i=1}$$

Only Γ_{x-q} depends on ψ_{xq}, we can therefore write it out to obtain

$$\begin{aligned} 0 \;&\approx\; 2\Gamma_{x2}\psi_{xq} + 2\Gamma_{x2+}\psi_{xq}\psi_x + 2g_{x-q} + \left[2g_{xq} + \Gamma_{x+q}\psi_x + \Gamma_{xx+}\left(I_n \otimes \psi_q\right) + \Gamma_{xx}\left(\psi_q \otimes I_n\right)\right]\psi_x + \\ &+\Gamma_{x-x+}\left(I_n \otimes \psi_q\right) + \Gamma_{x-x}\left(I_n \otimes \psi_q\right) + \left[E^i_t \left[w'_{t+1}\right] \operatorname{mat}_{k,n}\left(\Gamma^i_{x-w+}\right)\right]^n_{i=1} \end{aligned} \tag{50}$$

The coefficients on $[w_t]$, $[w_t \otimes w_{t+1}]$ and $[w_{t+1} \otimes w_t]$ determine the contribution of the first term in (46) to the determination of ψ_{wq}. The i-th equation for these terms reads

$$\Gamma^i_{wq} w_t + \tilde{E}^i_t \left[\Gamma^i_{w+w}\left(w_{t+1} \otimes w_t\right)\right] + \tilde{E}^i_t \left[\Gamma_{ww+}\left(w_t \otimes w_{t+1}\right)\right]$$

After reorganizing and stacking, we obtain

$$0 \approx \Gamma_{wq} + \left[\tilde{E}^i_t \left[w'_{t+1}\right] \left(\operatorname{mat}_{k,k}\left(\Gamma^i_{w+w}\right)\right)'\right]^n_{i=1} + \left[\tilde{E}^i_t \left[w'_{t+1}\right] \left(\operatorname{mat}_{k,k}\left(\Gamma^i_{ww+}\right)\right)\right]^n_{i=1}$$

Again, only the first term depends on ψ_{wq}, so that

$$\begin{aligned} 0 \;&\approx\; 2\Gamma_{x2}\psi_{wq} + 2g_{wq} + \Gamma_{x+w}\left(\psi_q \otimes I_k\right) + \Gamma_{xw}\left(\psi_q \otimes I_k\right) + \Gamma_{xq}\psi_w + \\ &+ \left[\tilde{E}^i_t \left[w'_{t+1}\right] \left(\operatorname{mat}_{k,k}\left(\Gamma^i_{w+w}\right)\right)'\right]^n_{i=1} + \left[\tilde{E}^i_t \left[w'_{t+1}\right] \left(\operatorname{mat}_{k,k}\left(\Gamma^i_{ww+}\right)\right)\right]^n_{i=1} \end{aligned} \tag{51}$$

Finally, the coefficients on $[w_{t+1}]$, $[w_{t+1} \otimes w_{t+1}]$ and the constant term will determine the contribution to ψ_{qq}:

$$0 \approx \Gamma^i_{qq} + \Gamma^i_{w+q}\tilde{E}^i_t \left[w_{t+1}\right] + \Gamma^i_{w+w+}\tilde{E}^i_t \left[w_{t+1} \otimes w_{t+1}\right]$$

which after stacking yields

$$0 \approx \Gamma_{qq} + \operatorname{diag}\left[\Gamma_{w+q}\tilde{\mathbb{E}}_t \left[w_{t+1}\right]\right] + \operatorname{diag}\left[\Gamma_{w+w+}\tilde{\mathbb{E}}_t \left[w_{t+1} \otimes w_{t+1}\right]\right]$$

Again, ψ_{qq} only shows up in the first term, so that

$$\begin{aligned} 0 \;&\approx\; \left(\Gamma_{x2+} + \Gamma_{x2}\right)\psi_{qq} + g_{qq} + \left(\Gamma_{x+q} + \Gamma_{xq}\right)\psi_q + \\ &+\operatorname{diag}\left[\Gamma_{w+q}\tilde{\mathbb{E}}_t \left[w_{t+1}\right] + \Gamma_{w+w+}\tilde{\mathbb{E}}_t \left[w_{t+1} \otimes w_{t+1}\right]\right]. \end{aligned} \tag{52}$$

Second term In order to compute $2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]$, first observe that

$$\begin{aligned} g_{1t+1} &= \left[\left(g_{x+}\psi_x + g_x\right)\psi_x + g_{x-}\right]x_{1t-1} + \left[\left(g_{x+}\psi_x + g_x\right)\psi_w + g_w\right]w_t + \\ &\quad + \left(g_{x+}\psi_x + g_{x+} + g_x\right)\psi_q + g_q + \left(g_{x+}\psi_w + g_{w+}\right)w_{t+1} = \\ &= \left(g_{x+}\psi_w + g_{w+}\right)w_{t+1} - \operatorname{diag}\left(\left(g_{x+}\psi_w + g_{w+}\right)\tilde{\mathbb{E}}_t\left[w_{t+1}\right]\right) \end{aligned}$$

where the terms drop out because of the relationships implied by (45).

From the solution for V^i_{2t+1} in Appendix C.1, we obtain

$$\begin{aligned} V^i_{2t+1} - \tilde{E}^i_t\left[V^i_{2t+1}\right] &= \left[2V^i_x\psi_{xw} + V^i_{xx}\left(\psi_x\otimes\psi_w\right)\right]\left[x_{1t}\otimes w_{t+1} - \tilde{E}^i_t\left[x_{1t}\otimes w_{t+1}\right]\right] + \\ &\quad + \left[V^i_{xx}\left(\psi_w\otimes\psi_x\right)\right]\left[w_{t+1}\otimes x_{1t} - \tilde{E}^i_t\left[w_{t+1}\otimes x_{1t}\right]\right] \\ &\quad + \left[2V^i_x\psi_{wq} + V^i_{xx}\left(\psi_w\otimes\psi_q + \psi_q\otimes\psi_w\right) + 2V^i_{xq}\psi_w\right]\left[w_{t+1} - \tilde{E}^i_t w_{t+1}\right] + \\ &\quad + \left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w\otimes\psi_w\right)\right]\left[w_{t+1}\otimes w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\otimes w_{t+1}\right]\right] \end{aligned} \tag{53}$$

This term is the crucial component of the worst-case belief distortion M^i_{1t+1} in (16). For the i-the equation in $2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]$, we therefore have:

$$2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]^i = -\frac{1}{\theta_i}\tilde{E}^i_t\left[\left(V^i_{2t+1} - \tilde{E}^i_t\left[V_{2t+1}\right]\right)\left(w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\right]\right)'\right]\left(g^i_{x+}\psi_w + g^i_{w+}\right)'$$

Substituting in the expression for $V^i_{2t+1} - \tilde{E}^i_t\left[V_{2t+1}\right]$, we need to compute the following terms:

$$\begin{aligned} \tilde{E}^i_t\left[\left(x_{1t}\otimes w_{t+1} - \tilde{E}^i_t\left[x_{1t}\otimes w_{t+1}\right]\right)\left(w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\right]\right)'\right] &= x_{1t}\otimes I_k \\ \tilde{E}^i_t\left[\left(w_{t+1}\otimes x_{1t} - \tilde{E}^i_t\left[w_{t+1}\otimes x_{1t}\right]\right)\left(w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\right]\right)'\right] &= I_k\otimes x_{1t} \\ \tilde{E}^i_t\left[\left(w_{t+1} - \tilde{E}^i_t w_{t+1}\right)\left(w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\right]\right)'\right] &= I_k \\ \tilde{E}^i_t\left[\left(w_{t+1}\otimes w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\otimes w_{t+1}\right]\right)\left(w_{t+1} - \tilde{E}^i_t\left[w_{t+1}\right]\right)'\right] &= I_k\otimes\tilde{E}^i_t\left[w_{t+1}\right] + \tilde{E}^i_t\left[w_{t+1}\right]\otimes I_k \end{aligned}$$

Thus

$$\begin{aligned} 2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]^i &= -\frac{1}{\theta_i}\left[2V^i_x\psi_{xw} + V^i_{xx}\left(\psi_x\otimes\psi_w\right)\right]\left(x_{1t}\otimes\left(g^i_{x+}\psi_w + g^i_{w+}\right)'\right) \\ &\quad -\frac{1}{\theta_i}\left[V^i_{xx}\left(\psi_w\otimes\psi_x\right)\right]\left(\left(g^i_{x+}\psi_w + g^i_{w+}\right)'\otimes x_{1t}\right) \\ &\quad -\frac{1}{\theta_i}\left[2V^i_x\psi_{wq} + V^i_{xx}\left(\psi_w\otimes\psi_q + \psi_q\otimes\psi_w\right) + 2V^i_{xq}\psi_w\right]\left(g^i_{x+}\psi_w + g^i_{w+}\right)' \\ &\quad -\frac{1}{\theta_i}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w\otimes\psi_w\right)\right]\left(\left(g^i_{x+}\psi_w + g^i_{w+}\right)'\otimes\tilde{E}^i_t\left[w_{t+1}\right]\right) \\ &\quad -\frac{1}{\theta_i}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w\otimes\psi_w\right)\right]\left(\tilde{E}^i_t\left[w_{t+1}\right]\otimes\left(g^i_{x+}\psi_w + g^i_{w+}\right)'\right) \end{aligned}$$

The bottom three lines contribute to the constant term in the set of equilibrium conditions, which in turn determines ψ_{qq}. The first two lines contribute through $x_{1t} = \psi_x x_{1t-1} + \psi_w w_t + \psi_q$ to the coefficients on x_{1t-1}, w_t and the constant. Rewrite the first two lines as

$$\begin{aligned}
&-\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right) \mathrm{mat}_{k,n}\left[2V^i_x\psi_{xw} + V^i_{xx}\left(\psi_x \otimes \psi_w\right)\right] x_{1t} \\
&-\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right)\left(\mathrm{mat}_{n,k}\left[V^i_{xx}\left(\psi_w \otimes \psi_x\right)\right]\right)' x_{1t}
\end{aligned}$$

which we can use to define $G^i_x x_{1t}$ with

$$G^i_x = -\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right)\left[\mathrm{mat}_{k,n}\left[2V^i_x\psi_{xw} + V^i_{xx}\left(\psi_x \otimes \psi_w\right)\right] + \left(\mathrm{mat}_{n,k}\left[V^i_{xx}\left(\psi_w \otimes \psi_x\right)\right]\right)'\right]$$

Then the contribution of $2E_t\left[\mathbb{M}_{1t+1}g_{1t+1}\right]$ to the individual terms in the set of equilibrium conditions is

$$\begin{aligned}
[x_{1t-1}] &: \left[G^i_x\right]^n_{i=1}\psi_x && (54)\\
[w_t] &: \left[G^i_x\right]^n_{i=1}\psi_w && (55)\\
[const] &: \left[G^i_x\right]^n_{i=1}\psi_q - && (56)\\
&-\left[\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right)\left[2V^i_x\psi_{wq} + V^i_{xx}\left(\psi_w \otimes \psi_q + \psi_q \otimes \psi_w\right) + 2V^i_{xq}\psi_w\right]'\right]^n_{i=1} \\
&-\left[\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right)\left(\mathrm{mat}_{k,k}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w \otimes \psi_w\right)\right]\right)'\tilde{E}^i_t\left[w_{t+1}\right]\right]^n_{i=1} \\
&-\left[\frac{1}{\theta_i}\left(g^i_{x+}\psi_w + g^i_{w+}\right)\left(\mathrm{mat}_{k,k}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w \otimes \psi_w\right)\right]\right)\tilde{E}^i_t\left[w_{t+1}\right]\right]^n_{i=1}
\end{aligned}$$

where the $[\ \cdot\ ]^n_{i=1}$ operator indicates horizontal stacking of rows.

Computational strategy for the second derivative We now put the contribution of the two terms in (46) together and device a strategy on how to co-determine ψ_{ij} and V^i_{ij}.

1. First compute ψ_{xx}, ψ_{xw} and ψ_{ww}. These are solely determined by the contribution of the first term in (46), and do not depend on the second derivatives of V^i. In order to do so, solve equation (47), (48), and (49).
2. The term ψ_{xx} is sufficient to determine V^i_{xx} using equation (42).
3. With V^i_{xx}, we can compute $\left[G^i_x\right]^n_{i=1}$ in (54), and thus also:
   - ψ_{xq} by adding terms in (50) and $\left[G^i_x\right]^n_{i=1}\psi_x$ from (54);
   - ψ_{wq} by adding terms in (51) and $\left[G^i_x\right]^n_{i=1}\psi_w$ from (55).
4. Compute V^i_{xq} in (42).

5. Now we can solve for ψ_{qq} using terms in (52) and adding the constant vector consisting of terms in (56).

This procedure completely determines all the belief distortions representing the worst-case models and the second-order law of motion for the state vector.

C.3 Distorted dynamics

The approximation we use to construct the dynamics under the worst-case model assumes that we replace M^i_{t+1} with an approximation that replaces V^i_{t+1} in the formula with its second-order approximation and do not expand M^i_{t+1} further:

$$M^i_{t+1} \approx \hat{M}^i_{t+1} \equiv \frac{\exp\left(-\frac{1}{\theta_i}\left(V^i_{1t+1} + \frac{1}{2}V^i_{2t+1}\right)\right)}{E_t\left[\exp\left(-\frac{1}{\theta_i}\left(V^i_{1t+1} + \frac{1}{2}V^i_{2t+1}\right)\right)\right]}$$

This expression is strictly positive and has a unitary mean, and it can therefore be used as a change of measure.[5] Also, since V^i_{1t+1} is linear in w_{t+1} and V^i_{2t+1} is quadratic, we can write it as

$$\hat{M}^i_{t+1} = \frac{\exp\left(\left(\hat{A}^i_0 + \hat{A}^i_1 x_{1t}\right)' w_{t+1} + \hat{B}^i\left(w_{t+1}\otimes w_{t+1}\right)\right)}{E_t\left[\exp\left(\left(\hat{A}^i_0 + \hat{A}^i_1 x_{1t}\right)' w_{t+1} + \hat{B}^i\left(w_{t+1}\otimes w_{t+1}\right)\right)\right]}$$

To derive $\hat{A}^i_0$, $\hat{A}^i_1$ and $\hat{B}^i$, use the solutions for V^i_{1t} and V^i_{2t} from Appendix C.1 and equation (53). Ignoring terms that are in time-t information set, we have

$$\begin{aligned}
-\frac{1}{\theta_i}\left(V^i_{1t+1} + \frac{1}{2}V^i_{2t+1}\right) \propto & -\frac{1}{\theta_i}\left[V^i_x\psi_w + V^i_x\psi_{wq} + \frac{1}{2}V^i_{xx}\left(\psi_w\otimes\psi_q + \psi_q\otimes\psi_w\right) + V^i_{xq}\psi_w\right] w_{t+1} \\
& -\frac{1}{\theta_i}\left(x_{1t}\right)'\left[\left(\operatorname{mat}_{k,n}\left[V^i_x\psi_{xw} + \frac{1}{2}V^i_{xx}\left(\psi_x\otimes\psi_w\right)\right]\right)' + \operatorname{mat}_{n,k}\left[\frac{1}{2}V^i_{xx}\left(\psi_w\otimes\psi_x\right)\right]\right] w_{t+1} \\
& -\frac{1}{2\theta_i}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w\otimes\psi_w\right)\right]\left[w_{t+1}\otimes w_{t+1}\right]
\end{aligned}$$

This immediately implies that

$$\begin{aligned}
\hat{A}^i_0 &= -\frac{1}{\theta_i}\left[V^i_x\psi_w + V^i_x\psi_{wq} + \frac{1}{2}V^i_{xx}\left(\psi_w\otimes\psi_q + \psi_q\otimes\psi_w\right) + V^i_{xq}\psi_w\right]' \\
\hat{A}^i_1 &= -\frac{1}{\theta_i}\left[\left(\operatorname{mat}_{k,n}\left[V^i_x\psi_{xw} + \frac{1}{2}V^i_{xx}\left(\psi_x\otimes\psi_w\right)\right]\right)' + \operatorname{mat}_{n,k}\left[\frac{1}{2}V^i_{xx}\left(\psi_w\otimes\psi_x\right)\right]\right]' \\
\hat{B}^i &= -\frac{1}{2\theta_i}\left[V^i_x\psi_{ww} + V^i_{xx}\left(\psi_w\otimes\psi_w\right)\right]
\end{aligned}$$

[5] Moreover, this expression corresponds to the first-order logarithmic expansion of M^i_{t+1}, compensated to make its mean equal to one. We choose this first-order expansion as an appropriate one here, since the second-order term in the expansion of M^i_{t+1} does not contribute to the solution of the second-order approximation of the model.

Utilizing formula (33), we deduce that under the distorted measure $\hat{\cdot}^i$, the shock w_{t+1} is distributed as $w_{t+1} \sim N\left(\hat{\mu}_t^i, \hat{\sigma}^i\left(\hat{\sigma}^i\right)'\right)$ where

$$\hat{\sigma}^i\left(\hat{\sigma}^i\right)' = \left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\hat{B}^i\right)\right]\right)^{-1} \tag{57}$$

$$\hat{\mu}^i = \hat{\sigma}^i\left(\hat{\sigma}^i\right)'\left(\hat{A}_0^i + \hat{A}_1^i x_{1t}\right) = \hat{\mu}_0^i + \hat{\mu}_1^i x_{1t} \tag{58}$$

The approximate distortion $\hat{M}_{t+1}^i$ therefore induces a time-varying change in the drift of the shock that is a linear function of the state vector x_{1t}, and a constant adjustment in its volatility.

We can therefore write

$$w_{t+1} = \hat{\mu}_0^i + \hat{\mu}_1^i x_{1t} + \hat{\sigma}^i \hat{w}_{t+1}^i$$

where $\hat{w}_{t+1}^i \sim N(0, I_k)$ under $\hat{\cdot}^i$. Under the distorted dynamics implied by $\hat{\cdot}^i$, the model behaves as

$$\begin{aligned}
x_{0t} &= \psi\left(x_{0t-1}, 0\right) \\
x_{1t} &= \left[\psi_x + \psi_w \hat{\mu}_1^i\right] x_{1t-1} + \psi_w \hat{\sigma}^i \hat{w}_t^i + \psi_w \hat{\mu}_0^i + \psi_q \\
x_{2t} &= \psi_x x_{2t-1} + \left[\psi_{xx} + 2\psi_{xw}\left(I_n \otimes \hat{\mu}_1^i\right) + \psi_{ww}\left(\hat{\mu}_1^i \otimes \hat{\mu}_1^i\right)\right]\left(x_{1t-1} \otimes x_{1t-1}\right) + \\
&\quad + \left[2\psi_{xw}\left(I_n \otimes \hat{\sigma}^i\right) + \psi_{ww}\left(\left(\hat{\mu}_1^i\right) \otimes \hat{\sigma}^i + \left[\hat{\sigma}^i \otimes \left(\hat{\mu}_1^i\right)_{\cdot j}\right]_{i=1}^k\right)\right]\left(x_{1t-1} \otimes \hat{w}_t^i\right) + \\
&\quad + \left[2\psi_{xq} + 2\psi_{xw}\left(I_n \otimes \hat{\mu}_0^i\right) + 2\psi_{wq}\hat{\mu}_1^i + \psi_{ww}\left(\hat{\mu}_0^i \otimes \hat{\mu}_1^i + \hat{\mu}_1^i \otimes \hat{\mu}_0^i\right)\right] x_{1t-1} + \\
&\quad + \psi_{ww}\left(\hat{\sigma}^i \otimes \hat{\sigma}^i\right)\left(\hat{w}_t^i \otimes \hat{w}_t^i\right) + \left[2\psi_{wq}\hat{\sigma}^i + \psi_{ww}\left(\hat{\mu}_0^i \otimes \hat{\sigma}^i + \hat{\sigma}^i \otimes \hat{\mu}_0^i\right)\right]\hat{w}_t^i + \\
&\quad + \psi_{qq} + \psi_{ww}\left(\hat{\mu}_0^i \otimes \hat{\mu}_0^i\right) + 2\psi_{wq}\hat{\mu}_0^i
\end{aligned}$$

D Static robust problems

The minimization problem (6) of the agent endowed with multiplier preferences leads to the first-order condition

$$0 = \log C + \theta\left(\log M + 1\right) + \kappa$$

where κ is the Lagrange multiplier on the constraint $E[M] = 1$. Since $E[M] = 1$, the solution for the worst-case distortion must necessarily be

$$M = \frac{\exp\left(-\frac{1}{\theta}\log C\right)}{E\left[\exp\left(-\frac{1}{\theta}\log C\right)\right]} = \exp\left(-\frac{1}{2\theta^2}\left(\mathsf{q}\sigma\right)^2 - \frac{1}{\theta}\mathsf{q}\sigma W\right) \tag{59}$$

Substituting this worst-case distortion into the objective function then yields

$$u^{mult} = -\theta \log E\left[\exp\left(-\frac{1}{\theta}\log C\right)\right] = \mu - \frac{1}{2}\left(\mathsf{q}\sigma\right)^2 - \frac{1}{2}\frac{1}{\theta}\left(\mathsf{q}\sigma\right)^2$$

which is the same objective as that of the power utility agent, u^{pow}, when $\theta = (\gamma - 1)^{-1}$. The two preference structures are thus isomorphic.

For the constraint preferences (7), write the Langrangean

$$L^{con} = E\left[M \log C\right] + \theta\left(E\left[M \log M\right] - \eta\right) + \kappa\left(E\left[M\right] - 1\right)$$

where θ is the endogenous Lagrange multiplier to be determined. The first-order condition implies the same form (59) for the worst-case distortion. In order to determine θ, compute

$$\eta = E\left[M \log M\right] = \frac{1}{2\theta^2}\left(\mathsf{q}\sigma\right)^2 \qquad \Longrightarrow \qquad \frac{1}{\theta} = \frac{\sqrt{2\eta}}{\mathsf{q}\sigma}$$

which implies result (9).

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