Existence and Uniqueness of Equilibrium Asset Prices over Infinite Horizons

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ABSTRACT. We obtain both necessary and sufficient conditions for existence and uniqueness of equilibrium asset prices in discrete-time, arbitrage free settings with dividend streams that have no natural termination date. We connect our conditions, and hence the problem of existence and uniqueness of asset prices, with the recent literature on stochastic discount factor decompositions using the principal eigenpairs of valuation operators. In addition, we show how local spectral radius theory can be used to obtain and interpret these principal eigenvalues.

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1. INTRODUCTION

One of the most fundamental problems in economics is the pricing of an asset paying a stochastic cash flow. In arbitrage-free discrete-time environments, the equilibrium price process $\{P_t\}_{t\geq 0}$ associated with a dividend process $\{D_t\}_{t\geq 1}$ must obey

$$P_t = \mathbb{E}_t M_{t+1}(P_{t+1} + D_{t+1}) \quad \text{for all } t \ge 0, \tag{1}$$

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where $\{M_t\}_{t \ge 1}$ is the pricing kernel or stochastic discount factor (SDF) process of a representative investor.¹ The two questions addressed in this paper are:

- 1. When can we associate to each pair $\{D_t, M_t\}_{t \ge 1}$ a unique and finite equilibrium price process $\{P_t\}_{t \ge 0}$?
- 2. How can we characterize and evaluate such prices when they exist?

The first question can be formulated more precisely as follows: If we regard (1) as a map from the joint distribution of $\{D_t, M_t\}$ to a price process $\{P_t\}$ satisfying (1), then when—and under what circumstances—is this mapping well defined? The second question concerns (a) existence and uniqueness of Markov equilibria and (b) connection to prices obtained by forward iteration.²

These issues are of ongoing concern for researchers who build asset pricing models, since existence, uniqueness and stability of equilibria are directly tied to the relative predictive power of their models (and since no one wishes to make statements about the implications of their models that are vacuous).³ At the same time, issues of existence and uniqueness have been sidelined in the applied literature in recent years because the majority of applied work has been conducted using approximation techniques like various types of perturbation methods, producing models that are easier to manipulate and interpret.

There is now, however, substantial evidence that these approximation methods may distort outcomes in ways that matter for quantification, interpretation and

¹Here and below, price is calculated for ex-dividend contracts. For overviews, see Kreps (1981), Cochrane and Hansen (1992), Hansen and Renault (2009) or Duffie (2010).

²Throughout the paper, we concern ourselves only with fundamental solutions rather than rational bubbles. For a recent discussion of the latter see Brunnermeier (2016).

³For example, consider the study of Epstein and Zin (1989), who write "…..we have not demonstrated the consistency of our analysis with a general equilibrium framework such as Lucas' (1978) stochastic pure endowment economy. Such an extension would need to confront the questions of existence and uniqueness of equilibrium asset prices. Moreover, Lucas' contraction mapping techniques would not suffice for the same reasons that those techniques were inadequate in establishing Theorem 3.1. Thus we leave such an extension to a separate paper." Three decades later this issue remains unresolved. The same is true for a variety of asset pricing models, with a range of SDF and dividend specifications.

prediction of asset prices. For example, Pohl et al. (2018) find that numerical approximations associated with Campbell–Shiller log-linearization significantly distort findings on risk premia and lead to large errors in measurements related to volatility and the price dividend ratio.

In addition to this demand for a better understanding of models that embed substantial nonlinearities, progress in understanding asset price dynamics over long horizons and their relationship to risk preferences has been made through new work on stochastic discount factor decompositions in Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Hansen (2012), Christensen (2017) and Qin and Linetsky (2017). These decompositions are used to extract a permanent growth component and a martingale component from the stochastic discount process, with the rate in the permanent growth component being driven by the principal eigenvalue of the corresponding valuation operator. While this literature uses the permanent growth component to gain insight on the structure of valuation for payoffs at alternative horizons, it is natural to ask how these same ideas might be applied to existence and uniqueness of equilibrium asset prices in infinite horizon economies.

In the present paper we connect this line of research on the study of principal eigenvalues of valuation operators to existence and uniqueness of equilibrium asset prices and price-dividend ratios in infinite horizon settings. We obtain both necessary and sufficient conditions for existence and uniqueness of equilibrium asset prices based around the principal eigenpairs of valuation operators. In addition, we show how local spectral radius theory can be used to obtain and interpret these principal eigenvalues.

To understand our methodology, a useful way to begin is to view the SDF M_{t+1} in (1), which serves to deflate payoffs in future states, as a random "contraction factor" around which a contraction mapping argument can be built, looking forwards in time. The operator in this contraction argument has as its fixed point an equilibrium price function, which is a map from the state process into an equilibrium price process. The operator itself is referred to below as the *equilibrium price operator*.

In the risk neutral case, where $M_{t+1} = \beta$ and $\beta < 1$, we have a uniform contraction rate of β in every state of the world, the equilibrium price operator is a contraction

of modulus β , and existence and uniqueness of the price process is immediate. Outside of this case, however, the SDF is random and $M_{t+1} > 1$ usually holds on a set of positive probability—since payoffs in bad states are highly valued. Contraction based arguments must confront this positive probability of expansion.

An early example of a successful study in this context is Lucas (1978), who uses a change-of-variable argument to remove the stochastic component from the contraction coefficient in his model. The model in question has SDF

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)},$$
(2)

where $\{C_t\}$ is a stationary consumption process, β is a state independent discount component and u is a period utility function. By solving for $u'(C_t)P_t$ instead of P_t directly, Lucas (1978) obtains a modified pricing operator with contraction modulus equal to β .

This methodology can be generalized. For example, while Lucas (1978) assumes that dividends are stationary, one can make a similar argument in the case where dividend growth is stationary instead (see, e.g., Mehra and Prescott (2003)). Also, while Lucas (1978) requires that utility and dividends are bounded, so that the equilibrium price operator acts in a space of continuous bounded functions, similar results can be obtained in unbounded settings by truncating innovations or working with weighting functions and weighted supremum norms (see, e.g., Alvarez and Jermann (2005) or Brogueira and Schütze (2017)). Finally, the change-of-variable technique works not just for SDFs of the form (2), but also for any SDF that can be decomposed into the product of a state independent discount factor and a ratio of stationary factors.

Unfortunately, for many SDFs used in modern asset pricing applications, no such decomposition exists and the change-of-variable technique fails. Examples include those found in Epstein and Zin (1989), Bansal and Yaron (2004) and Schorfheide et al. (2018). Moreover, a substantial body of evidence shows that SDFs that possess a stationary factorization of the form (2) *cannot* match asset price data in several important dimensions (see, in particular, Borovička et al. (2016)).

When factoring M_{t+1} is not possible, one can still consider treating the entire SDF as a "random contraction factor." For example, even if $M_{t+1} > 1$ holds with positive probability, a contraction argument can still be constructed if, say, $\mathbb{E}_t M_{t+1} < 1$

in "most" states. This is the approach taken in Calin et al. (2005), who obtain existence and uniqueness results for equilibrium prices for a class of models involving habit formation by using contraction arguments in a space of integrable functions.

However, there is a common problem with all of the approaches to existence and uniqueness of asset prices listed above: they require contraction *in one step*. This condition is too tight, in the sense that it excludes many stable models with well defined equilibrium prices. In fact modern asset pricing models aimed at quantitative applications often select parameterizations close to the boundary between stability and instability (see, e.g., sections 5.2–6.1). In such settings, conditions based on one step contractions fail and provide no useful information.

Here we adopt an alternative approach that considers instead discounting over the long run, using the implied *n* period state price deflator $\prod_{i=1}^{n} M_{t+i}$ with large *n*, and requiring only that contraction occurs "on average, eventually." For example, we show that, assuming a stationary dividend process and some basic regularity on the structure of the problem, the equilibrium price operator is a L_1 contraction at *some* finite power whenever

$$\lim_{n \to \infty} \left\{ \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n} < 1.$$
(3)

From this we obtain existence and uniqueness of equilibrium asset prices. (When dividend *growth* is stationary, rather than dividends, condition (3) is modified to feature a dividend-growth adjusted SDF.)

Results for elementary cases are easily recovered from condition (3). For example, in the risk neutral case $M_t = \beta$, the left hand side of (3) is just β . If $\{M_t\}$ is random but IID, then the limit in (3) is $\mathbb{E}M_t$ and hence $\mathbb{E}M_t < 1$ is sufficient for stability. Moreover, for transition independent SDFs such as (2), intermediate terms cancel when we take the product in (3), leading to simple conditions that recover (and extend) classical results.

For more complex SDFs, such as those arising from recursive preferences or habit formation, (3) can be evaluated either analytically, by calculating expectations and taking limits, or numerically, analogous to the way that spectral radius conditions of finite dimensional systems are examined in order to test stability. Examples of the analytical approach are given in sections 5.1 and 6.1. The numerical approach

is used to show equilibrium asset prices exist and are unique for the Epstein–Zin specifications adopted in Bansal and Yaron (2004) and Schorfheide et al. (2018). The numerical implementations exploit the fact that the expression in (3) is easily approximated by Monte Carlo when $\{M_t\}$ can be simulated. This calculation turns out to be stable and accurate even for moderate simulation runs (see section 5.1).

The above discussion corresponds to one special case of our results, where the underlying function space is L_1 . We also study outcomes in other function spaces, the benefit of which is that varying the function space introduces the possibility of imposing additional structure on the solution to the problem, such as continuity, or finiteness of second moments. Working in an abstract setting that includes such function spaces, we show that existence and uniqueness of equilibrium asset prices hold whenever r(V) < 1, where r(V) is the spectral radius of the valuation operator V that maps future payoffs to current values via the set of state price deflators embedded in the stochastic discount factor. The L_1 results are a special case because, as we show using local spectral radius results, the limit in (3) is equal to r(V) when the function space is L_1 .

By using long run average contractions as determined by the spectral radius of the valuation operator, we obtain conditions that are very close to necessary—in contrast to the conditions based on one step contractions discussed above. For example, we show via an application of the Krein–Rutman theorem that if the valuation operator V is also compact, then r(V) > 1 implies that T^n is not a contraction for any n, and, more importantly, that no equilibrium price function exists. We also use local spectral radius conditions to weaken the compactness requirement in the case of L_1 , since compactness is relatively stringent in this case.

The rest of our paper is structured as follows. Section 2 sets up a general version of the problem. Section 3 states our main results. Applications are treated in sections 4 and 5. Proofs are deferred to the appendix.

2. Preliminaries

In this section we set out the existence and uniqueness problems for asset prices considered in the paper.

2.1. Forward Looking Recursions. Consider the forward looking model

$$Y_t = \mathbb{E}_t \left[\Phi_{t+1}(Y_{t+1} + G_{t+1}) \right] \quad \text{for all } t \ge 0, \tag{4}$$

where { Φ_t } and { G_t } are given nonnegative stochastic processes and { Y_t } is endogenous. One version of (4) is the equilibrium price problem (1), where { Y_t } is the price process, { G_t } is cash flow and { Φ_t } is the SDF. Another version arises when dividends are nonstationary, in which case dividing (1) by D_t transforms the endogenous variable into the price-dividend ratio $Y_t = P_t/D_t$, with $G_{t+1} = 1$ and $\Phi_{t+1} = M_{t+1}D_{t+1}/D_t$.

We assume that $\{\Phi_t\}$ and $\{G_t\}$ admit the representations

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) \quad \text{and} \quad G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1}) \tag{5}$$

where { X_t } is an underlying state process, { η_t } is a W-valued innovation sequence and ϕ and g are Borel measurable maps from X × X × W to \mathbb{R}_+ . The sets X and W are arbitrary Polish spaces.⁴ The state process is assumed to be stationary and Markovian. The representations in (5) replicate the general multiplicative functional specifications considered in Hansen and Scheinkman (2009) and Hansen (2012), and are sufficient for all problems we consider.

A stochastic process $\{Y_t\}$ is called a *solution* of the pricing recursion (4) if it is nonnegative, finite \mathbb{P} -almost everywhere and (4) holds \mathbb{P} -almost surely.

Forward iteration yields the candidate solution

$$Y_t^F := \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^n \Phi_{t+i} G_{t+n} \right], \tag{6}$$

which states that current price equals current expectation of total lifetime cash flow appropriately discounted. While (6) can be understood intuitively as the "fundamental solution," we refer to it for now as the *forward projection*, since it does not yet meet our definition of a solution (it could for example be infinite).

In what follows, the Borel sets of X are denoted by \mathscr{B} and the stochastic kernel generating $\{X_t\}$ is denoted by Π .⁵ The process $\{X_t\}$ is defined on some underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and satisfies $\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \Pi(x, B)$

⁴The Polish assumption, which requires that X and W are separable and completely metrizable, is very weak and satisfied in all applications of which we are aware.

⁵In particular, Π is a function from (X, \mathscr{B}) to [0, 1] such that $B \mapsto \Pi(x, B)$ is a probability measure on (X, \mathscr{B}) for each $x \in X$, and $x \mapsto \Pi(x, B)$ is \mathscr{B} -measurable for each $B \in \mathscr{B}$.

for all $x \in X$ and $B \in \mathscr{B}$. The innovation process $\{\eta_t\}$ is assumed to be IID and independent of $\{X_t\}$. Each η_t has common distribution ν . The common marginal distribution of each X_t is denoted by π . Relations such as (4) and (6) are understood as \mathbb{P} -almost sure equalities, while conditional expectations are with respect to the natural filtration generated by $\{X_t\}$.

2.2. **Markov Solutions.** Given the assumptions on the exogenous processes made in (5), it is natural to seek a Markov solution $Y_t^* = h^*(X_t)$ to the recursive pricing equation (4), where h^* is a fixed function in some candidate set \mathcal{H} . We require \mathcal{H} to be a Banach lattice of real-valued functions on X with the usual notions of pointwise order, addition and scalar multiplication.⁶ Let $\|\cdot\|$ denote the norm on \mathcal{H} and let \leq denote the partial order. Let \mathcal{H}_+ denote the *positive cone* of \mathcal{H} , consisting of all functions in \mathcal{H} taking only nonnegative values.

As usual, the *norm* of a bounded linear operator L from \mathcal{H} to itself is defined as

$$||L|| := \sup_{\|f\|=1} ||Lf||.$$

The *spectrum* $\sigma(L)$ of *L* is all scalars $\lambda \in \mathbb{C}$ such that $L - \lambda I$ fails to be bijective. A scalar λ is called an *eigenvalue* of *L* if there exists a nonzero $f \in \mathcal{H}$ such that $Lf = \lambda f$. The function *f* is then called an *eigenfunction*. The set $\sigma(L)$ is nonempty and compact in the complex plane, and every eigenvalue of *L* lies in $\sigma(L)$. The *spectral radius* of *L* is $r(L) := \max\{|\lambda| : \lambda \in \sigma(L)\}$. Since \mathcal{H} is a Banach space, Gelfand's formula holds:

$$r(L) = \lim_{n \to \infty} \|L^n\|^{1/n}.$$
 (7)

The operator *L* is called *compact* if the image under *L* of the unit ball in \mathcal{H} lies in a compact subset of \mathcal{H} . *L* is called *positive* if it maps the positive cone \mathcal{H}_+ into itself.

In what follows, the notion of quasi-interiority will play a significant role, allowing us to tie spectral radius conditions to local spectral radius results, thereby sharpening our findings and allowing us to obtain clean representations of our stability

⁶A Banach lattice is a Riesz space that is also complete. To accommodate unbounded solutions, the set \mathcal{H} will in some instances be identified with an L_p space. Elements of \mathcal{H} are then equivalence classes of functions, rather than functions *per se* and pointwise statements such as equalities and inequalities are understood as almost everywhere requirements.

conditions. To define this notion of interiority, let \mathcal{L} be all continuous linear functionals $\ell : \mathcal{H} \to \mathbb{R}$ such that $\ell(h) \ge 0$ whenever $h \in \mathcal{H}_+$. A function $h \in \mathcal{H}_+$ is called *quasi-interior* to \mathcal{H}_+ if $\ell(h) > 0$ for every nonzero $\ell \in \mathcal{L}$.

Example 2.1. If X is compact and \mathcal{H} is $\mathscr{C}(X)$, the set of continuous functions on X endowed with the supremum norm, then \mathcal{H} is a Banach lattice with the property that any strictly positive function in $\mathscr{C}(X)$ is both interior to the positive cone and also quasi-interior.⁷

Example 2.2. Let π be the common marginal distribution of each X_t , as above, let p be a constant satisfying $p \ge 1$ and let $\mathcal{H} = L_p(\pi)$, the space of Borel measurable functions $g: X \to \mathbb{R}$ such that

$$||h|| := \left\{ \int |h(x)|^p \pi(\mathrm{d}x) \right\}^{1/p}$$
(8)

is finite. Functions equal π -almost everywhere are identified, so that (8) defines a norm on $L_p(\pi)$ and together with the pointwise order they form a Banach lattice. While the positive cone of $L_p(\pi)$ contains no interior points, every strictly positive function in $L_p(\pi)$ is quasi-interior.⁸

In the remainder of the paper, \mathcal{H} will be one of the function spaces in examples 2.1– 2.2. The L_p setting is more general than that of $\mathscr{C}(X)$, since every bounded measurable function on X lies in $L_p(\pi)$. Moreover, every $L_p(\pi)$ spaces lies in $L_1(\pi)$.⁹

Our findings on necessary conditions and principle eigenvalues of valuation operators rely in part on a result, due to Zabreiko et al. (1967) and Mirosława Zima (private communication), concerning the local spectral radii of positive compact linear operators acting on quasi-interior points:

⁷ Let $h \in \mathscr{C}(X)$ be strictly positive on X. By the Riesz–Markov–Kakutani representation theorem, for each $\ell \in \mathcal{L}$, there is regular Borel measure μ on X such that $\ell(h) = \int h \, d\mu$ for each $h \in \mathscr{C}(X)$. If μ is not the zero measure, then $\ell(h) = \int h \, d\mu > 0$. Hence h is quasi-interior.

⁸Let $h \in L_p(\pi)$ be strictly positive π -almost everywhere on X, and let q be such that 1/p + 1/q = 1. By the Riesz representation theorem, given nonzero $\ell \in \mathcal{L}$, there is a $g \in L_q(\pi)$ such that $\ell(h) = \int gh \, d\pi$, where g is positive on a set of positive π measure. Evidently gh is likewise positive on a set of positive π measure, and hence $\ell(h) = \int gh \, d\pi > 0$. Hence h is quasi-interior.

⁹While $L_1(\pi)$ is therefore the most general setting, the more specialized spaces give additional structure, as discussed in the introduction (for example, Markov solution in $L_2(\pi)$ have finite second moment, while solutions in $\mathscr{C}(X)$ are continuous). Hence we present most of our theory in the setting of a generic Banach lattice \mathcal{H} .

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Theorem 2.1 (Zabreiko–Krasnosel'skii–Stetsenko–Zima). *Let h be an element of* \mathcal{H}_+ *and let* $L: \mathcal{H} \to \mathcal{H}$ *be a compact linear operator. If* L *is also positive and h is quasi-interior, then*

$$\lim_{n \to \infty} \|L^n h\|^{1/n} = r(L).$$
(9)

In general, $r(h, L) := \limsup_{n\to\infty} \|L^n h\|^{1/n}$ is called the *local spectral radius* of *L* at *h*. Under the conditions of theorem 2.1, the limit supremum equals the limit, which in turn equals the spectral radius (cf., Gelfand's formula in (7)). While a number of related results can be obtained, theorem 2.1 is particularly useful because it allows us to consider spaces where the positive cone has empty interior (e.g., the L_p spaces in example 2.2). As we could find no complete proof published in English we include one in the appendix.

3. Results

In this section we set out the core theoretical results of the paper.

3.1. The Equilibrium Price Operator. Any Markov solution $h^* \in \mathcal{H}$ to the price recursion (4) must satisfy

$$h^{*}(x) = \int \int \phi(x, x', \eta) \left[h^{*}(x') + g(x, x', \eta) \right] \nu(\mathrm{d}\eta) \Pi(x, \mathrm{d}x')$$
(10)

for all $x \in X$. In other words, h^* is a fixed point of the *equilibrium price operator* T defined at $h \in \mathcal{H}$ by

$$Th = Vh + \hat{g},\tag{11}$$

where

$$Vh(x) := \int h(x') \left[\int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}x')$$
(12)

and

$$\hat{g}(x) := \int \int \phi(x, x', \eta) g(x, x', \eta) \nu(\mathrm{d}\eta) \Pi(x, \mathrm{d}x').$$
(13)

We call *V* the *valuation operator* by analogy with the asset pricing models described above.

Assumption 3.1. Together, *V*, \hat{g} and \mathcal{H} have the following properties:

- (a) The integral $\int |h| d\pi$ is finite for all $h \in \mathcal{H}$.
- (b) The valuation operator *V* maps \mathcal{H} to itself and \hat{g} is in \mathcal{H} .

Part (a) of assumption 3.1 means that any Markov solution $Y_t = h(X_t)$ with $h \in \mathcal{H}$ will have finite first moment. Weakening this assumption would imply substantial technical difficulties, since the problem definition embedded in the forward looking restriction (4) is stated in terms of conditional expectations, and conditional expectations are themselves defined in terms of unconditional expectations. Part (b) of assumption 3.1 implies that *V* is a nontrivial positive linear operator on \mathcal{H} (nontrivial because we are assuming an arbitrage free environment) and the equilibrium price operator *T* is a self-mapping both on \mathcal{H} and \mathcal{H}_+ . Any fixed point of *T* in \mathcal{H}_+ is called an *equilibrium price function*.

Theorem 3.1. *Let assumption 3.1 hold. If the valuation operator V is compact, then the following statements are equivalent:*

- (a) r(V) < 1.
- (b) There exists an $n \in \mathbb{N}$ such that T^n is a contraction on $(\mathcal{H}, \|\cdot\|)$.
- (c) There exists a unique equilibrium price function h^* in \mathcal{H}_+ such that $\lim_{n\to\infty} T^n h = h^*$ for every $h \in \mathcal{H}_+$.

That (a) implies (b) follows directly from Gelfand's formula. The proof that (b) implies (c) uses an extension of the Banach contraction mapping theorem, which implies existence of a unique, globally attracting fixed point of *T* in \mathcal{H} when (b) holds. Since \mathcal{H} is a Banach lattice, the set \mathcal{H}_+ is closed in \mathcal{H} and, given that *T* is a self-mapping on \mathcal{H}_+ , any fixed point must lie in \mathcal{H}_+ . Compactness of *V* is not required either of these steps. On the other hand, the fact that (c) implies (a) requires compactness of the operator because it exploits the Krein–Rutman theorem.

Theorem 3.1 opens a number of questions. First, in the stable case r(V) < 1, what is the connection between the fixed point of *T* and the forward projection? Second, in the unstable case $r(V) \ge 1$, does failure of contractivity imply failure of existence? Third, how can one evaluate the spectral radius conveniently? We turn to these issues below.

3.2. Further Results for the Stable Case. By theorem 3.1, the equilibrium price operator *T* has a unique, globally attracting fixed point h^* in \mathcal{H}_+ whenever r(V) < 1. The following theorem provides additional information. Note that compactness of *V* is not assumed.

Theorem 3.2. *If assumption* 3.1 *holds and* r(V) < 1*, then*

- (a) The unique equilibrium price function h^* is equal to $\sum_{n \ge 0} V^n \hat{g}$.
- (b) The process $\{Y_t^*\}$ defined by $Y_t^* := h^*(X_t)$ for all t solves (4).
- (c) The forward projection Y_t^F in (6) is finite and equal to Y_t^* with probability one.
- (d) If, in addition, $\lim_{n\to\infty} \int |h_n| d\pi = 0$ whenever $\{h_n\} \subset \mathcal{H}$ and $\lim_{n\to\infty} ||h_n|| = 0$, then no stationary Markov solution aside from $\{Y_t^*\}$ exists.

Part (a) of theorem 3.2 is an immediate consequence of the Neumann series theorem. Parts (b) and (c) use the Markov property of $\{X_t\}$ and assumption 3.1. The statement in part (d) that no other stationary Markov solution exists means that if $\{Y_t\}$ satisfies (4) and $\{Y_t\} = \{h(X_t)\}$ for some $h \in \mathcal{H}$, then $\{Y_t\}$ and $\{Y_t^*\}$ are *indistinguishable*. That is,

$$\mathbb{P}\{Y_t = Y_t^* \text{ for all } t\} = 1.$$
(14)

The condition on the norm of \mathcal{H} in (d) says that convergence in this norm is stronger than convergence in $L_1(\pi)$. It is satisfied in the settings of examples 2.1–2.2. It can be further improved in some environments by dropping the Markov restriction. The appendix gives one example (see proposition 8.1).

3.3. Further Results for the Unstable Case. What happens when r(V) exceeds unity? We know that T^n is not a contraction on \mathcal{H} for any $n \in \mathbb{N}$ from theorem 3.1, and that the statement in (c) of theorem 3.1 fails. However, this implies neither absence nor multiplicity of fixed points. Moreover, it is not immediately obvious that the results in theorem 3.2 fail when $r(V) \ge 1$. To see why, consider the equilibrium price function $h^* = \sum_{n=0}^{\infty} V^n \hat{g}$ from (a) of theorem 3.2. Even if r(V) > 1, the operator V can still have some eigenvalues with modulus strictly less than unity. If \hat{g} lies in a space spanned by the corresponding eigenfunctions, then the expression $\sum_{n=0}^{\infty} V^n \hat{g}$ can be well defined.

Nevertheless, r(V) > 1 does indeed imply divergence, as well as absence of a positive fixed point, when some regularity conditions are imposed. The next theorem gives details.

Theorem 3.3. Let \mathcal{H} be such that all strictly positive elements are quasi-interior to \mathcal{H}_+ . If, in addition, V is compact and r(V) > 1, then no equilibrium price function exists. The intuition behind this result is that \hat{g} is positive by assumption and hence the dynamics of the iterates $V^n \hat{g}$ are similar to the dynamics of $V^n e$, where e is the principal eigenfunction (which is also positive by the Krein–Rutman theorem whenever the latter holds). If r(V) > 1, then $V^n e$ diverges. Hence $T^n e$ diverges. The connection between the dynamics of $V^n \hat{g}$ and dynamics of $V^n e$ are formalized using the local spectral radius result in theorem 2.1.

3.4. Further Results for Integrable Functions. In this section we specialize to the case $\mathcal{H} = L_1(\pi)$ from example 2.2. This setting is important because the space $L_1(\pi)$ is large. For example, it allows us to tackle settings where the dividend process is unbounded, as is commonly assumed in applications. Moreover, as shown below, the local spectral radius condition yields a particularly simple expression for the spectral radius of the valuation operator *V* when we specialize to $L_1(\pi)$.

Note that $L_1(\pi)$ is not reflexive whenever the state space is infinite, and hence conditions for compactness of operators are stringent—and typically difficult to verify. This is potentially problematic when we wish to apply theorem 3.1 or theorem 3.3 because of the compactness requirement on *V*. In order to weaken this requirement, we introduce the following assumption:

Assumption 3.2. The state space X is endowed with some σ -finite Borel measure μ and the stochastic kernel Π for the state process has a density kernel $\pi(\cdot | \cdot)$ with respect to μ .¹⁰ Moreover, the function $\psi \colon X \to \mathbb{R}$ defined by

$$\psi(x) := \sup_{x' \in \mathsf{X}} \left\{ \int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \cdot \frac{\pi(x' \mid x)}{\pi(x')} \right\}$$
(15)

satisfies $\int \psi \, d\pi < \infty$.

Typically, X will be a subset of \mathbb{R}^d and μ is either Lebesgue measure or the counting measure. Below we show that assumption 3.2 is satisfied in some standard asset pricing applications.

As discussed in the appendix, assumption 3.2 implies that the valuation operator V is continuous as a linear operator on $L_1(\pi)$. In fact, when (15) is valid, V is a Hille– Tamarkin operator on $L_1(\pi)$, which we show is sufficient to obtain an L_1 version of the local spectral radius result in theorem 2.1 without imposing compactness:

¹⁰That is, π is a real-valued Borel measurable map on X × X satisfying $\Pi(x, B) = \int_B \pi(x' \mid x) \mu(dx')$ for every $B \in \mathscr{B}$.

Proposition 3.4. If assumption 3.2 holds, then the sequence $\{r_{\Phi}^n\}$ defined by

$$r_{\Phi}^{n} := \left\{ \mathbb{E} \prod_{i=1}^{n} \Phi_{i} \right\}^{1/n}$$
(16)

converges to r(V) as $n \to \infty$.

In fact the limit of r_{Φ}^n is precisely $r(\mathbb{1}_X, V)$, the local spectral radius of the valuation operator at $\mathbb{1}_X$, the function identical to unity on X. The proof of proposition 3.4 takes the expression $\lim_{n\to\infty} ||L^nh||^{1/n}$ from (9) and then replaces L with V, h with $\mathbb{1}_X$ and $|| \cdot ||$ with the L_1 norm. It then uses the law of iterated expectations to simplify the resulting expression. This iterated expectation step works because, for positive elements of $L_1(\pi)$, the L_1 norm is additive.

By proposition 3.4, if assumption 3.2 holds and $\lim_{n\to\infty} r_{\Phi}^n < 1$, then all the stability results in theorem 3.2 hold with $\mathcal{H} = L_1(\pi)$. This verifies one of the claims put forward in the introduction.

The next proposition shows that the compactness condition used to study the unstable case r(V) > 1 in theorem 3.3 can be weakened to assumption 3.2 when we are working in L_1 . It also strengthens the nonexistence result in theorem 3.3 beyond Markov solutions. The proof uses of proposition 3.5 uses proposition 3.4.

Proposition 3.5. If assumption 3.2 holds and r(V) > 1, then no equilibrium price function exists in L_1 . Moreover, the recursive pricing equation (4) has no stationary solution with finite first moment.

3.5. **Calculating the Spectral Radius.** In terms of computation, one benefit of the result in (16) is that it allows us to approximate r(V) via Monte Carlo. In particular, we have

$$\lim_{m \to \infty} \left\{ \frac{1}{m} \sum_{j=1}^{m} \prod_{i=1}^{n} \Phi_i^{(j)} \right\}^{1/n} = r_{\Phi}^n \quad \mathbb{P}\text{-almost surely,}$$
(17)

where each $\Phi_1^{(j)}, \ldots, \Phi_n^{(j)}$ is an independently simulated path of $\{\Phi_t\}$. This follows from the strong law of large numbers combined with the fact that $Z_n \to Z \mathbb{P}$ -a.s. implies $g(Z_n) \to g(Z) \mathbb{P}$ -a.s. whenever $g \colon \mathbb{R} \to \mathbb{R}$ is continuous.

The benefit of using (17) to calculate r(V) is that one needs only to be able to simulate the (growth adjusted) stochastic discount factor process. Such a calculation

avoids discretization and is highly parallelizable, since each path is simulated independently. It is accurate in the applications we consider with only moderate sample sizes (see, e.g., tables 1–2 on page 20, which compares the Monte Carlo calculations with the true value of the spectral radius calculated analytically).

A second point regarding computation of r(V) is that, even in a setting where \mathcal{H} is some Banach lattice other than $L_1(\pi)$, the result in proposition 3.4 will be valid for any numerical implementation. The reason is that, in numerical calculations, X is always a finite subset of the floating point numbers, in which case \mathcal{H} , as a Banach lattice of real-valued functions defined on a finite set, is isomorphic and strongly equivalent to \mathbb{R}^d endowed with the norm topology, where *d* is the number of elements in the finite set. In particular, with $d := \operatorname{card} X$, each function in \mathcal{H} is identified with a unique vector in \mathbb{R}^d and any two norms on a finite dimensional Banach space are strongly equivalent.¹¹ It then follows from Gelfand's formula that the spectral radius of *V* is identical, regardless of which norm we use. Hence it suffices to calculate the spectral radius in the $L_1(\pi)$ setting. Moreover, assumption 3.2 is always satisfied when X is finite, so proposition 3.4 applies.

4. APPLICATIONS PART I: STATIONARY DIVIDENDS

We now turn to applications, focusing in this section on some well known models where dividends themselves are required to be stationary. (In section 5 we consider more empirically plausible assumptions.) The space of candidate solutions will be $L_1(\pi)$, where, as before, π is the marginal distribution of the state process $\{X_t\}$.

4.1. **Bounded Utility.** Consider the asset pricing problem of Lucas (1978), where the price process obeys (1) and the stochastic discount factor is

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}.$$
(18)

Here C_t is consumption, β is a discount factor, and u is utility. Our first goal is to recover the existence and uniqueness result for prices obtained in Lucas (1978) using theorem 3.2.

¹¹Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space *E* are strongly equivalent if there exist strictly positive constants *k* and ℓ such that $k\|x\| \leq \|x\|' \leq \ell \|x\|$ for all *x* in *E*.

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To match the result in Lucas (1978), we take $Y_t := P_t u'(C_t)$ to be the endogenous object rather than P_t , where the latter represents the price of a claim to a stationary dividend stream { D_t }. Equation (1) yields

$$Y_t = \mathbb{E}_t \left[\beta(Y_{t+1} + u'(C_{t+1})D_{t+1}) \right].$$
(19)

In equilibrium, $C_t = D_t = d(X_t)$ for all t, where $\{X_t\}$ is the Markov state process and d is a given function. Equation (19) is a version of (4) with $\Phi_t = \beta$ and $G_{t+1} = u'(D_{t+1})D_{t+1}$. The functions ϕ and g in (5) are therefore

$$\phi(x, x', \eta) := \beta$$
 and $g(x, x', \eta) := u'(d(x'))d(x').$ (20)

The function \hat{g} defined in (13) becomes

$$\hat{g}(x) = \beta \int u'(d(x'))d(x')\Pi(x, dx').$$
 (21)

Lucas (1978) assumes that u is concave and bounded, which in turn gives $0 \le u'(d(x'))d(x') \le N$ for some $N \in \mathbb{N}$. Hence \hat{g} is bounded by βN , and therefore an element of $L_1(\pi)$. Moreover, $Vg = \beta g$ for any $g \in L_1(\pi)$, so V maps $L_1(\pi)$ to itself, implying that assumption 3.1 holds.

Moreover, the fact that $Vg = \beta g$ for any $g \in L_1(\pi)$ implies that the range space of *V* is one dimensional. In particular, the only eigenvalue of *V* is β , and hence the spectral radius r(V) of *V* is also equal to β . (One can also obtain the same conclusion by observing that, since $\Phi_t = \beta$, the expression on the right hand side of (16) is equal to β for all *n*.) Theorem 3.2 then implies the existence of a unique stationary Markov equilibrium whenever $\beta < 1$. This is the same conclusion as proposition 3 of Lucas (1978).

4.2. **Constant Relative Risk Aversion.** The previous result relies on boundedness of utility, an assumption that is rarely satisfied in applications. We can drop this assumption provided that $\hat{g} \in L_1(\pi)$ continues to hold true. To give one example, consider the work of Brogueira and Schütze (2017), who use a weighted sup norm approach to extend the results of Lucas (1978) to the case $u(c) = c^{1-\gamma}/(1-\gamma)$ with $d(x) = \exp(x)$ and $\{X_t\}$ following

$$X_{t+1} = \rho X_t + b + \sigma \xi_{t+1}, \quad \{\xi_t\} \stackrel{\text{ind}}{\sim} N(0,1) \quad \text{and} \quad |\rho| < 1,$$
(22)

....

In this case the definitions of ϕ and g in (20) are unchanged, while \hat{g} in (21) becomes

$$\hat{g}(x) = \beta \exp\left\{ (1-\gamma) \left(\rho x + b + \frac{(1-\gamma)\sigma^2}{2} \right) \right\}.$$
(23)

Since π is Gaussian, we have $\hat{g} \in L_1(\pi)$. The conditions of theorem 3.2 are again satisfied and hence a uniquely defined Markov solution $Y_t^* = h^*(X_t)$ exists. This recovers the main result of Brogueira and Schütze (2017) without their requirement of a positively correlated state process and several additional parameter restrictions. We can of course go further, dropping the AR(1) assumption and modifying the utility and dividend process specifications, provided that $\hat{g}(X_t)$ still has a finite first moment.

5. APPLICATIONS PART II: STATIONARY DIVIDEND GROWTH

The standard theory discussed in the previous section takes dividends to be stationary. Such models can be brought closer to the data by assuming instead that dividend growth is stationary. In this case we aim to solve for the price-dividend ratio $Q_t := P_t/D_t$, which, in view of (1), must satisfy

$$Q_t = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} (Q_{t+1} + 1) \right].$$
 (24)

Let us summarize the implications of the preceding results for the solution of the price-dividend ratio Q_t in (24). Comparing (24) and (4), in this context we have

$$\Phi_{t+1} = \phi(X_t, X_{t+1}, \eta_{t+1}) = M_{t+1} \frac{D_{t+1}}{D_t}$$
(25)

and $G_{t+1} = g(X_t, X_{t+1}, \eta_{t+1}) = 1$. Let

$$r_M := \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{i=1}^n M_i \frac{D_n}{D_0} \right\}^{1/n}.$$
 (26)

whenever the limit exists. The next result summarizes the L_1 theory of section 3.4 in terms of its implications for (24), the forward looking recursion for the pricedividend ratio.

Proposition 5.1. *If the conditions of assumption 3.2 hold, then the limit in* (26) *is welldefined and finite. Moreover,*

- (a) $r_M = \lim_{n\to\infty} r_{\Phi}^n = r(V)$, where r_{Φ}^n is as defined in (16) and V is the spectral radius of the valuation operator associated with (25).
- (b) If $r_M < 1$, then a unique stationary Markov solution $Q_t^* = h^*(X_t)$ for (24) exists, where $h^* \in L_1(\pi)$. In particular, the conclusions of theorem 3.2 are valid.
- (c) Conversely, if $r_M > 1$, then the price-dividend ratio equation (24) has no stationary solution with finite first moment.

In the rest of this section we connect these results to several applications.

5.1. **CRRA Utility and Stochastic Dividend Growth.** Consider a benchmark asset pricing model as found in, say, Mehra and Prescott (2003), where $\ln D_{t+1} - \ln D_t = X_{t+1}$ for some stationary Markov process $\{X_t\}$. With $C_t = D_t$ and CRRA utility, this yields

$$\Phi_{t+1} := M_{t+1} \frac{D_{t+1}}{D_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} = \beta \exp\left\{(1-\gamma)X_{t+1}\right\}.$$
 (27)

Hence $\phi(x, x', \eta) = \beta \exp((1 - \gamma)x')$. Let $\{X_t\}$ follow the AR(1) process in (22).

Consider first assumption 3.2. Connecting the definition of ψ in that assumption to the present application, we have

$$\psi(x) \propto \sup_{x' \in \mathbb{R}} \exp\left\{ (1-\gamma)x' - \frac{(x'-\rho x-b)^2}{2\sigma^2} + \frac{(x'-\mu_s)^2}{2\sigma_s^2} \right\}.$$
 (28)

Here \propto means "proportional to," μ_s is the stationary mean $b/(1-\rho)$ and σ_s^2 is the stationary variance $\sigma^2/(1-\rho^2)$. The stationary variance is larger than the conditional variance σ^2 , so the supremum in (28) is finite. Simple arguments show that, after substituting the maximizing value of x' into the right hand side of (28), we have $\psi(x) \propto \exp(a_0 + a_1x + a_2x^2)$ for suitable constants a_i . As the stationary distribution of a Gaussian AR(1) process, π is itself Gaussian, and hence $\int \psi \, d\pi$ is finite. In particular, assumption 3.2 and the conditions of proposition 5.1 hold.

As a consequence, existence of a finite price-dividend ratio depends on

$$r_M = \lim_{n \to \infty} \left\{ \mathbb{E} \prod_{i=1}^n \Phi_i \right\}^{1/n} = \beta \lim_{n \to \infty} \exp \left\{ (1-\gamma) \sum_{i=1}^n X_i \right\}^{1/n},$$

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where the second equality is due to (27). Since $\{X_t\}$ obeys (22), we have

$$\exp\left\{(1-\gamma)\sum_{i=1}^{n} X_{i}\right\}^{1/n} = \exp\left\{(1-\gamma)\frac{\mu_{n}}{n} + \frac{(1-\gamma)^{2}s_{n}^{2}}{2n}\right\}$$

where μ_n is the expectation of $\sum_{i=1}^{n} X_i$ and s_n^2 is its variance. Elementary manipulations yield

$$\frac{s_n^2}{n} = \frac{\sigma^2}{1-\rho^2} \left\{ 1 + \frac{2(n-1)}{n} \frac{\rho}{1-\rho} - \frac{2\rho^2}{n} \frac{1-\rho^{n-1}}{(1-\rho)^2} \right\}.$$

Hence

$$r_M = r(V) = \beta \exp\left\{ (1 - \gamma) \left[\frac{b}{1 - \rho} + \frac{1 - \gamma}{2} \frac{\sigma^2}{(1 - \rho)^2} \right] \right\}.$$
 (29)

Proposition 5.1 implies that a unique solution with finite first moment exists—and equals the forward projection—whenever (29) evaluates to strictly less than unity. If, on the other hand, $r_M > 1$, then no such solution exists.

The spectral radius r(V) in (29) represents the discounted risk-adjusted growth rate of aggregate consumption C_t , and reveals the dual role parameter γ plays under CRRA utility. The term in brackets is the average risk-adjusted consumption growth rate. The rest of the expression constitutes intertemporal discounting, consisting of the time-preference parameter β and the contribution of intertemporal substitution captured by the term $1 - \gamma$ multiplying the average growth rate.

Since an analytical expression for the spectral radius exists, the current setting provides a useful test case for the proposal to calculate the spectral radius of the valuation operator using Monte Carlo, via (17). Our interest is in examining whether or not the Monte Carlo based expressions are sufficiently accurate for moderate sample sizes. Tables 1–2 are supportive. The parameters here are chosen to match Mehra and Prescott (2003), with $\beta = 0.99$, $\rho = 0.941$, $\gamma = 2.5$, $\sigma = 0.000425$ and b = 0.00104 in table 1, while in table 2 we shifted γ to 2.0. The actual value of r(V) indicated in the table caption is calculated from the closed form expression (29). The interpretation of n and m in the table is consistent with the left hand side of (17). In both tables the approximation is accurate up to five decimal places in all simulations.

5.2. Long Run Risk Part I. Next we turn to an asset pricing model with Epstein– Zin utility and stochastic volatility in cash flow and consumption estimated by

TABLE 1. Monte Carlo spectral radius estimates when r(V) = 0.9659169

	n = 400	n = 600	n = 800	n = 1000	n = 1200
m = 10000	0.9659158	0.9659146	0.9659107	0.9659181	0.9659126
m = 15000	0.9659137	0.9659146	0.9659154	0.9659193	0.9659161
m = 20000	0.9659119	0.9659144	0.9659159	0.9659166	0.9659142
m = 25000	0.9659167	0.9659127	0.9659155	0.9659161	0.9659127

TABLE 2. Monte Carlo spectral radius estimates when r(V) = 0.9727279

	n = 400	n = 600	n = 800	n = 1000	n = 1200
m = 10000	0.9727255	0.9727252	0.9727256	0.9727278	0.9727246
m = 15000	0.9727223	0.9727223	0.9727268	0.9727274	0.9727263
m = 20000	0.9727269	0.9727253	0.9727275	0.9727265	0.9727267
m = 25000	0.9727247	0.9727267	0.9727260	0.9727267	0.9727278

Bansal and Yaron (2004). Preferences are represented by the continuation value recursion

$$V_{t} = \left[(1 - \beta) C_{t}^{1 - 1/\psi} + \beta \left\{ \mathcal{R}_{t} \left(V_{t+1} \right) \right\}^{1 - 1/\psi} \right]^{1/(1 - 1/\psi)},$$
(30)

where $\{C_t\}$ is the consumption path extending on from time *t* and

$$\mathcal{R}_t(V_{t+1}) := (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}.$$
(31)

The parameter $\beta \in (0,1)$ is a time discount factor, γ governs risk aversion and ψ is the elasticity of intertemporal substitution. Dividends and consumption grow according to

$$g_{t+1}^{c} = \mu_{c} + z_{t} + \sigma_{t} \eta_{c,t+1},$$

$$g_{t+1}^{d} = \mu_{d} + \alpha z_{t} + \phi_{d} \sigma_{t} \eta_{d,t+1},$$

$$z_{t+1} = \rho z_{t} + \phi_{z} \sigma_{t} \eta_{z,t+1},$$

$$\sigma_{t+1}^{2} = \max \left\{ v \sigma_{t}^{2} + d + \phi_{\sigma} \eta_{\sigma,t+1}, 0 \right\},$$

where $g_{t+1}^d = \ln(D_{t+1}/D_t)$ and $g_{t+1}^c = \ln(C_{t+1}/C_t)$. Here $\{\eta_{i,t}\}$ are IID and standard normal for $i \in \{d, c, z, \sigma\}$. The state X_t can be represented as $X_t = (z_t, \sigma_t)$. The SDF associated with this model is

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1}\right)^{\theta - 1}$$

where W_t is the aggregate wealth-consumption ratio and $\theta := (1 - \gamma)/(1 - 1/\psi)$. See, for example, Bansal and Yaron (2004), p. 1503. Hence

$$\Phi_{t+1} := M_{t+1} \frac{D_{t+1}}{D_t} = \beta^{\theta} \frac{D_{t+1}}{D_t} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{W_{t+1}}{W_t - 1}\right)^{\theta - 1}.$$
(32)

To obtain the value of the aggregate wealth-consumption ratio $\{W_t\}$ we exploit the fact that $W_t = w(X_t)$ where *w* solves the Euler equation

$$\beta^{\theta} \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^{\theta} \right] = 1.$$

Rearranging and using the expression for consumption growth given above, this equality can be expressed as

$$w(z,\sigma) = 1 + [Kw^{\theta}(z,\sigma)]^{1/\theta},$$

where *K* is the operator

$$Kg(z,\sigma) = \beta^{\theta} \exp\left\{ (1-\gamma)(\mu_c + z) + \frac{(1-\gamma)^2 \sigma^2}{2} \right\} \Pi g(z,\sigma)$$
(33)

In this expression, $\Pi g(z, \sigma)$ is the expectation of $g(z_{t+1}, \sigma_{t+1})$ given the state's law of motion, conditional on $(z_t, \sigma_t) = (z, \sigma)$.

The existence of a unique solution $w = w^*$ to (5.2) in the set of continuous functions $\mathscr{C}(X)$ under the parameterization used in Bansal and Yaron (2004) is established in Borovička and Stachurski (2017) when the innovation terms $\{\eta_{i,t}\}$ are truncated, so that the state space is compact. To use this result, we study the same setting and seek an equilibrium price-dividend ratio function in $\mathscr{C}(X)$. In particular, we compute w^* using the iterative method described in Borovička and Stachurski (2017), recover W_t as $w^*(X_t)$ for each t and evaluate Φ_{t+1} via (32). We use this to compute the spectral radius r(V) of the valuation operator V.

As discussed in section 3.5, to approximate r(V) in the present context, we can use the Monte Carlo average in (17). In computing the product $\prod_{t=1}^{n} \Phi_t$ we express it

$$\begin{split} \prod_{t=1}^{n} \Phi_t &= (\beta^{\theta} \exp(\mu_d - \gamma \mu_c))^n \\ &\times \exp\left((\alpha - \gamma) \sum_{t=1}^{n} z_t - \gamma \sum_{t=1}^{n} \sigma_t \eta_{c,t+1} + \phi_d \sum_{t=1}^{n} \sigma_t \eta_{d,t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t \right), \end{split}$$

where $\hat{w}_{t+1} = \ln[W_{t+1}/(W_t - 1)]$. We generate this value *m* times, average and raise to the power of 1/n to obtain the approximation of the spectral radius in (17). Our implementation uses a JIT-compiled and parallelized implementation based on Numba, which runs on a regular workstation in around 3 seconds when n = 5,000 and m = 10,000.

At the parameter values using in Bansal and Yaron (2004), we find that r(V) = 0.9969, implying the existence of a unique equilibrium price-dividend ratio function in $\mathscr{C}(X)$.¹² While this value is close to 1, significant shifts in parameters are required to cross the boundary r(V) = 1. For example, figure 1 shows the spectral radius r(V) calculated at a range of parameter values in the neighborhood of the Bansal and Yaron (2004) specification via a contour map. The parameter α is varied on the horizontal axis, while μ_d is on the vertical axis. Other parameters are held fixed at the Bansal and Yaron (2004) values. The black contour line shows the boundary between stability and instability. Instability (and absence of a finite solution) is associated with high mean dividend growth μ_d and low coefficient α on the low frequency persistent component of dividend growth.

5.3. Long Run Risk Part II. Now we repeat the exercise in section 5.2 but using instead the dynamics for consumption and dividends in Schorfheide et al. (2018),

as

¹²Following Bansal and Yaron (2004), the parameters are $\gamma = 10.0$, $\beta = 0.998$, $\psi = 1.5 \mu_c = 0.0015$, $\rho = 0.979$, $\phi_z = 0.044$, v = 0.987, d = 7.9092e-7, $\phi_\sigma = 2.3e$ -6. $\mu_d = 0.0015$, $\alpha = 3.0$ and $\phi_d = 4.5$. See table IV on page 1489. The values of *n* and *m* in (17) in this calculation were set to 1,000 and 10,000 respectively. Although a range of alternative values were tested, none changed the main conclusion.



FIGURE 1. The spectral radius r(V) in the Bansal–Yaron model

which are given by

$$g_{t+1}^{c} = \mu_{c} + z_{t} + \sigma_{c,t} \eta_{c,t+1},$$

$$g_{t+1}^{c} = \mu_{d} + \alpha z_{t} + \delta \sigma_{c,t} \eta_{c,t+1} + \sigma_{d,t} \eta_{d,t+1},$$

$$z_{t+1} = \rho z_{t} + (1 - \rho^{2})^{1/2} \sigma_{z,t} v_{t+1},$$

$$\sigma_{i,t} = \varphi_{i} \bar{\sigma} \exp(h_{i,t}),$$

$$h_{i,t+1} = \rho_{h_{i}} h_{i} + \sigma_{h_{i}} \xi_{i,t+1}, \quad i \in \{z, c, d\}.$$

The innovation vectors $\eta_t = (\eta_{c,t}, \eta_{d,t})$ and $\xi_t := (v_t, \xi_{z,t}, \xi_{c,t}, \xi_{d,t})$ are IID over time, mutually independent and standard normal in \mathbb{R}^2 and \mathbb{R}^4 respectively. The state can be represented as the four dimensional vector

$$X_t := (z_t, h_{z,t}, h_{c,t}, h_{d,t}).$$
(34)

Otherwise the analysis and methodology for computing the spectral radius is similar to section 5.2. The product of growth adjusted stochastic discount factors over



FIGURE 2. The spectral radius r(V) in the Schorfheide–Song–Yaron model

n period from t = 1 is

$$\prod_{t=1}^{n} \Phi_t = (\beta^{\theta} \exp(\mu_d - \gamma \mu_c))^n$$
$$\exp\left((\alpha - \gamma) \sum_{t=1}^{n} z_t + (\delta - \gamma) \sum_{t=1}^{n} \sigma_{c,t} \eta_{c,t+1} + \sum_{t=1}^{n} \sigma_{d,t} \eta_{d,t+1} + (\theta - 1) \sum_{t=1}^{n} \hat{w}_t\right)$$

As before, we generate this product m times and then average to obtain the approximation of the spectral radius in (17).

Figure 2 shows the spectral radius r(V) calculated at a range of parameter values in the neighborhood of the Schorfheide et al. (2018) specification via a contour map. The parameter φ_d is varied on the horizontal axis, while μ_d is on the vertical axis. Other parameters are held fixed at the Schorfheide et al. (2018) values. As before, the black contour line shows the boundary between stability and instability.

6. APPLICATIONS PART III: COMPARISON WITH ALTERNATIVES

In this section we investigate how the spectral radius condition for existence and uniqueness of asset prices constructed above compare to alternative conditions based on one step contractions. Our results indicate that the conditions provided by these one step methods are too strict to be useful in modern quantitative applications.

6.1. **Habit Persistence.** In models with consumption externalities such as those found in Abel (1990) and Campbell and Cochrane (1999), SDFs have the form

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{s(H_t)}{s(H_0)},$$
(35)

where *s* is a given function and H_t is the ratio of consumption to a social stock of past and present consumption. In Abel (1990), and in particular in the case of "external" habit formation, the price dividend ratio implied by this stochastic discount factor satisfies the forward recursion (24) with

$$M_{t+1}\frac{D_{t+1}}{D_t} = k_0 \exp((1-\gamma)(\rho - \alpha)X_t)$$
(36)

where $k_0 := \beta \exp(b(1-\gamma) + \sigma^2(\gamma-1)^2/2)$ and α is a preference parameter. The connection between (35) and (36) is detailed in section 2.1 of Calin et al. (2005). The state sequence $\{X_t\}$ obeys (22) with $b := x_0 + \sigma^2(1-\gamma)$. Here x_0 is a parameter indicating mean constant growth rate of the dividend of the asset. See Calin et al. (2005) for details. In our notation,

$$\phi(x, x', \eta) = k_0 \exp((1-\gamma)(\rho - \alpha)x). \tag{37}$$

We wish to exploit the results in proposition 5.1, which necessitates checking condition (15). We have

$$\psi(x) \propto \sup_{x' \in \mathbb{R}} \exp\left\{ (1-\gamma)(\rho-\alpha)x + \frac{-(x'-\rho x-b)^2}{2\sigma^2} + \frac{(x'-\mu_s)^2}{2\sigma_s^2} \right\}$$

Similar analysis to that conducted in section 5.1 shows that ψ is in $L_1(\pi)$, and hence the conditions of proposition 5.1 hold.

The spectral radius can also be computed in similar fashion to section 5.1, yielding

$$r(V) = k_0 \exp\left((1-\gamma)(\rho-\alpha)\frac{b}{1-\rho} + \frac{(1-\gamma)^2(\rho-\alpha)^2}{2}\frac{\sigma^2}{(1-\rho)^2}\right).$$

By proposition 5.1, a unique solution with finite first moment exists whenever r(V) < 1 and fails to exist when r(V) > 1.

To give some basis for comparison, let us contrast the condition r(V) < 1 with the sufficient condition for existence and uniqueness of an equilibrium price-dividend



FIGURE 3. One step test value and spectral radius r(V) with $x_0 = 0.05$

ratio found in proposition 1 of Calin et al. (2005), which implies a one step contraction. Since the spectral radius condition requires only eventual contraction and is almost necessary we can expect it to be weaker than the condition of Calin et al. (2005).

Figure 3 supports this conjecture. Each sub-figure shows the results of either the one step or the spectral radius test at a range of parameter values. The left sub-figure shows the one step test values obtained by evaluating the expression in equation (7) of Calin et al. (2005). The right sub-figure gives the spectral radius r(V). The horizontal and vertical axes show grid points for the parameters β and σ respectively. Pairs (β , σ) with test values strictly less than one (points to the south west of the 1.0 contour line) are where the respective condition holds. Points to the north east of this contour line are where it fails. Inspection of the figure shows that the sufficient condition in Calin et al. (2005) requires an unrealistic discount factor, and fails for many parameterizations that do in fact have unique stationary Markov equilibria.¹³

6.2. **One Step Contractions in the Long Run Risk Model.** As discussed in the introduction, most of the effort in existing literature has gone toward providing conditions under which the equilibrium price operator is a contraction in one step, which allows for easy application of Banach's fixed point theorem. The preceding

¹³The parameters held fixed in figure 3 are, following Calin et al. (2005), $\rho = 0.98$, $\gamma = 2.5$, $x_0 = 0.05$ and $\alpha = 1$.

section showed that this approach is problematic for a habit formation model. Let us now consider the same scenario for the long run risk model of Bansal and Yaron (2004).

We know from section 5.2 that the model and parameterization used in Bansal and Yaron (2004) is stable, with a unique equilibrium price function in the positive cone of $\mathscr{C}(X)$ for the asset corresponding to their dividend process. This was established by showing that the equilibrium price operator is *eventually* contracting on average, since the spectral radius r(V) is less than unity. Here we show that one step contraction does not lead to the same conclusion. Although our results are tied to the specific contraction coefficient we develop, when combined with similar findings from section 6.1, they suggest that methodologies built around one step contractions will be ineffectual when studying realistic asset pricing problems.

To begin, consider the setting of section 5.2, and let

$$\kappa(V) = \sup_{x \in \mathsf{X}} \int \int \phi(x, x', \eta) \Pi(x, \mathrm{d}x) \nu(\mathrm{d}\eta).$$

Observe that, for any two functions *h* and *h'* in $\mathscr{C}(X)$ and any point $x \in X$,

$$|Th(x) - Th'(x)| = \left| \int (h(y) - h'(y)) \left[\int \phi(x, y, \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}y) \right| \\ \leq \int |h(y) - h'(y)| \left[\int \phi(x, y, \eta) \nu(\mathrm{d}\eta) \right] \Pi(x, \mathrm{d}y)$$

Using the definition of $\kappa(V)$ and letting $\|\cdot\|$ be the supremum norm in $\mathscr{C}(X)$, we then have $|Th(x) - Th'(x)| \leq \kappa(V) \|h - h'\|$. Taking the supremum over $x \in X$, this yields the bound

$$||Th - Th'|| \leq \kappa(V)||h - h'||.$$

Thus $\kappa(V)$ provides a natural contraction coefficient, in the sense that if $\kappa(V) < 1$, then *T* is a (one step) contraction mapping.

Figure 4 shows $\kappa(V)$ for the Bansal–Yaron model over a range of parameters, with the black contour line indicating the boundary between satisfying and failing the condition $\kappa(V) < 1$. Evidently the Bansal–Yaron parameterization fails this criterion by a wide margin. We repeated the exercise for the parameterization in Schorfheide et al. (2018) and found a similar outcome.



FIGURE 4. The one step coefficient $\kappa(V)$ in the Bansal–Yaron model

7. CONCLUSION

In this paper we studied existence and uniqueness of equilibrium asset prices in discrete-time infinite horizon settings. We obtained both necessary and sufficient conditions for existence and uniqueness by connecting with the recent literature on stochastic discount factor decompositions based around principal eigenvalues and eigenfunctions of valuation operators. We showed how local spectral radius theory can be used to calculate the principal eigenvalues. This procedure allowed us to demonstrate existence and uniqueness of asset prices in well known applications where such fundamental properties were yet to be established.

We also found that for realistic asset pricing models, one step contraction conditions, which have been the focus of earlier studies on existence and uniqueness of equilibrium asset prices, typically fail at all empirically plausible parameterizations. While this result is hardly definitive—since there may be other norms and other contraction coefficients that enlarge the stable domain—the margin by which the Bansal–Yaron parameterization fails the test and the unrealistic parameter values required for the test to pass suggest that future research should concentrate on spectral radius methods rather than the one step contractions.

8. Appendix

Remaining proofs are completed below.

Proof of theorem 3.1. To see that (a) implies (b), suppose that r(V) < 1. Using Gelfand's formula, choose $n \in \mathbb{N}$ such that $||V^n|| < 1$. Then, for any $h, h' \in \mathcal{H}$ we have

$$||T^nh - T^nh'|| = ||V^nh - V^nh'|| = ||V^n(h - h')|| \le ||V^n|| \cdot ||h - h'||.$$

To go from (b) to (c), observe that \mathcal{H}_+ is closed in \mathcal{H} , since \mathcal{H} is a Banach lattice. Since *V* is positive, it maps \mathcal{H}_+ to itself. The remaining results follow from a wellknown extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)).

To show (c) implies (a), we will make use of the Krein–Rutman theorem applied to the operator *V*. In doing so, we note that the positive cone of \mathcal{H} is reproducing in \mathcal{H} , since every $h \in \mathcal{H}$ can be expressed as the difference between max{h, 0} and $-\min{\{h, 0\}}$. These functions lie in \mathcal{H} by the Banach lattice assumption and are obviously nonnegative.

Now suppose that (a) fails, so $r(V) \ge 1$. Since *V* is both positive and compact, the Krein–Rutman theorem (see theorem 41.2 of Zaanen (1997)) implies existence of an eigenfunction *e* such that Ve = r(V)e. Hence

$$||T^n 0 - T^n e|| = ||V^n 0 - V^n e|| = ||0 - r(V)^n e|| = r(V)^n ||0 - e||$$

As *e* is an eigenfunction it must be nonzero, so we have two points in \mathcal{H}_+ such that the distance between them fails to converge to zero. Such sequences cannot converge to the same point, so (c) cannot hold.

Proof of theorem 3.2. Let *I* denote the identity map on \mathcal{H} . Since r(V) < 1, there exists an $i \in \mathbb{N}$ such that $||V^i|| < 1$. As \mathcal{H} is a Banach space, the Neumann series theorem then implies that $(I - V)^{-1}$ is well-defined on \mathcal{H} and equals $\sum_{i=0}^{\infty} V^i$ (see, e.g., theorem 2.3.1 and corollary 2.3.3 of Atkinson and Han (2009)). In particular, $h^* = \sum_{n=0}^{\infty} V^n \hat{g}$ is a well-defined element of \mathcal{H} (using assumption 3.1, which gives $\hat{g} \in \mathcal{H}$). Moreover, $h^* = (I - V)^{-1} \hat{g}$ and hence

$$h^* = Vh^* + \hat{g}.$$
 (38)

Part (a) is now established.

Regarding (b), let $\{Y_t^*\}$ be defined by $Y_t^* = h^*(X_t)$ for all t. To show that this process solves (4), we need to show that it is nonnegative, almost everywhere finite and satisfies (4) with probability one. The first two claims follow immediately from $Y_t^* = \int h^*(X_t)$ and our assumptions on \mathcal{H}_+ . Regarding the third, observe that, for any fixed $t \in \mathbb{Z}$, we have

$$\mathbb{E}_{t} \left[\Phi_{t+1}(Y_{t+1}^{*} + G_{t+1}) \right] = \mathbb{E}_{t} \left\{ \phi(X_{t}, X_{t+1}, \eta_{t+1}) [Y_{t+1}^{*} + g(X_{t}, X_{t+1}, \eta_{t+1})] \right\}$$
$$= \int \int \phi(X_{t}, x', \eta) h^{*}(x') \nu(d\eta) \Pi(X_{t}, dx') + \hat{g}(X_{t})$$
$$= V h^{*}(X_{t}) + \hat{g}(X_{t})$$

In view of (38), this last expression evaluates to $h^*(X_t) = Y_t^*$. Thus, $\{Y_t^*\}$ satisfies (4), and claim (b) is established.

Regarding (c), as a first step we show that

$$V^{n-1}\hat{g}(X_t) = \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i} G_{t+n}$$
(39)

with probability one for all $n \in \mathbb{N}$. To see this, consider first the case n = 1. By the definition of \hat{g} we have

$$V^{0}\hat{g}(X_{t}) = \hat{g}(X_{t}) = \int \int \phi(X_{t}, x', \eta)g(X_{t}, x', \eta)\nu(\mathrm{d}\eta)\Pi(X_{t}, \mathrm{d}x')$$

= $\mathbb{E}_{t}\phi(X_{t}, X_{t+1}, \eta_{t+1})g(X_{t}, X_{t+1}, \eta_{t+1}) = \mathbb{E}_{t}\Phi_{t+1}G_{t+1}.$

Thus, (39) holds when n = 1. Now suppose it holds at arbitrary $n \in \mathbb{N}$. We claim it also holds at n + 1. Indeed,

$$V^{n}\hat{g}(X_{t}) = \int \int \phi(X_{t}, x', \eta) V^{n-1}\hat{g}(x')\nu(\mathrm{d}\eta)\Pi(X_{t}, \mathrm{d}x')$$

= $\mathbb{E}_{t} \phi(X_{t}, X_{t+1}, \eta_{t+1}) V^{n-1}\hat{g}(X_{t+1})$

Using the induction hypothesis and the law of iterated expectations,

$$V^n \hat{g}(X_t) = \mathbb{E}_t \Phi_{t+1} \mathbb{E}_{t+1} \prod_{i=2}^n \Phi_{t+i} G_{t+n} = \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i} G_{t+n}.$$

Thus, (39) holds for all n.

To complete the proof of (c), we use $h^* = \sum_{n \ge 0} V^n \hat{g}$ and (39) to obtain

$$h^*(X_t) = \sum_{n=1}^{\infty} V^{n-1} \hat{g}(X_t) = \sum_{n=1}^{\infty} \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i} G_{t+n} = \mathbb{E}_t \left[\sum_{n=1}^{\infty} \prod_{i=1}^n \Phi_{t+i} G_{t+n} \right]$$

The last equality in the previous display follows from the monotone convergence theorem.

Thus, $h^*(X_t)$ is indeed equal almost surely to Y_t^F in (6). The forward projection is finite almost surely because of this equality and the finite first moment of $h^*(X_t)$, which was proved in part (b).

Regarding part (d) of theorem 3.2, let $\{Y_t\}$ be a sequence satisfying both (4) and $Y_t = h(X_t)$ for some $h \in \mathcal{H}$. Forward iteration then gives

$$Y_t - Y_t^* = \mathbb{E}_t \left[\prod_{i=1}^n \Phi_{t+i} [h(X_{t+n}) - h^*(X_{t+n})] \right]$$

This is equivalent to

$$Y_t - Y_t^* = V^n q(X_t)$$
 where $q := h - h^*$

Thus, for any $n \in \mathbb{N}$ we have

$$\mathbb{E}|Y_t - Y_t^*| = \mathbb{E}|V^n q(X_t)|$$

By the spectral radius condition r(V) < 1 we have $||V^n q|| \le ||V^n|| ||q|| \to 0$ as $n \to \infty$. By the properties on the norm $|| \cdot ||$ imposed in the statement of claim (d), this yields $\mathbb{E}|V^n q(X_t)| \to 0$ as $n \to \infty$, from which we conclude that $\mathbb{E}|Y_t - Y_t^*| = 0$, and hence $\mathbb{P}\{Y_t = Y_t^*\} = 1$. Since the intersection of countable many probability one sets has probability one, the statement in (14) is established. Claim (d) is thus verified and the proof of theorem 3.2 is complete.

Proposition 8.1. If $\mathcal{H} = L_2(\pi)$ and the conditions of theorem 3.2 hold, then any stationary solution to (4) with finite second moment is indistinguishable from $\{Y_t^*\}$.

Proof of proposition 8.1. Let $\{Y_t\}$ be a stationary solution to (4) with finite second moment. Fixing *t* and iterating on (4) yields

$$Y_t = \mathbb{E}_t \left[\sum_{j=1}^n \prod_{i=1}^j \Phi_{t+i} G_{t+j} + \prod_{i=1}^n \Phi_{t+i} Y_{t+n} \right] \quad \text{for any } n \in \mathbb{N}.$$

Subtracting the analogous expression for Y_t^* gives

$$\mathbb{E}|Y_t - Y_t^*| \leq \mathbb{E}\left[\prod_{i=1}^n \Phi_{t+i} \cdot |Y_{t+n} - Y_{t+n}^*|\right].$$

From this bound and the Cauchy–Schwarz inequality we have

$$\mathbb{E}|Y_t - Y_t^*| \leqslant \sqrt{\mathbb{E}\left[\prod_{i=1}^n \Phi_{t+i}^2\right]} \mathbb{E}(Y_{t+n} - Y_{t+n}^*)^2.$$

By assumption, both $\{Y_t\}$ and $\{Y_t^*\}$ are stationary and have finite second moments. Moreover, by the definition of V we have $\mathbb{E} \prod_{i=1}^n \Phi_{t+i}^2 = ||V^n \mathbb{1}||^2$ where $\mathbb{1} \in L_2(\pi)$ is unity everywhere on X and $|| \cdot ||$ is the $L_2(\pi)$ norm. Since $||V^n \mathbb{1}|| =$ $||V^n|| \to 0$ as $n \to \infty$ by the spectral radius assumption, we conclude that $\mathbb{E}|Y_t - Y_t^*| = 0$. Hence $Y_t = Y_t^*$ with probability one. Since the time index is countable, it follows that $\{Y_t\}$ and $\{Y_t^*\}$ are indistinguishable, as was to be shown.

In the following result we use the fact that the space $(\mathcal{H}, \|\cdot\|)$ is assumed to be a Banach lattice when endowed with the pointwise order, which implies that the positive cone (the functions in \mathcal{H} taking nonnegative values) is both normal and reproducing.¹⁴

Proof of theorem 2.1. Let *h* and *L* be as in the statement of the theorem and let \mathcal{H}_+ be the positive cone of \mathcal{H} . Recall that $r(h, L) = \limsup_{n \to \infty} \|L^n h\|^{1/n}$ is the local spectral radius of *L* at *h*. From the definition of r(L) it suffices to show that $r(h, L) \ge r(L)$. To this end, let λ be a constant satisfying $\lambda > r(h, L)$ and let

$$h_{\lambda} := \sum_{n=0}^{\infty} \frac{L^n h}{\lambda^{n+1}}.$$
(40)

The point h_{λ} is a well-defined element of \mathcal{H}_+ by $\limsup_{n\to\infty} \|L^n h\|^{1/n} < \lambda$ and Cauchy's root test for convergence. It is also quasi-interior, since the sum in (40) includes the quasi-interior element h, and since L maps \mathcal{H}_+ into itself. Moreover, by the standard Neumann series theory of linear equations (e.g., Krasnosel'skii

¹⁴The positive cone of a partially ordered normed linear space is called *reproducing* if its linear span equals the whole space. It is called *normal* if there exists a constant *N* such that $||g|| \leq N||h||$ whenever $0 \leq g \leq h$.

et al. (2012), theorem 5.1), the point h_{λ} also has the representation $h_{\lambda} = (\lambda I - L)^{-1}h$, from which we obtain $\lambda h_{\lambda} - Lh_{\lambda} = h$. Because $h \in \mathcal{H}_+$, this implies that

$$Lh_{\lambda} \leqslant \lambda h_{\lambda}.$$
 (41)

Applying inequality (41), compactness of *L*, quasi-interiority of h_{λ} and theorem 5.5 (a) of Krasnosel'skii et al. (2012), we must have $r(L) \leq \lambda$. Since this inequality was established for an arbitrary λ satisfying $\lambda > r(h, L)$, we conclude that $r(h, L) \geq r(L)$.

Proof of theorem 3.3. Let \mathcal{H} and V have the stated properties and suppose that $h \in \mathcal{H}_+$ and h solves the functional equation (10), which is to say that $h = Vh + \hat{g}$. Iterating on this equation, we have

$$h = \hat{g} + V\hat{g} + \dots + V^n\hat{g} + V^{n+1}h$$

Since \mathcal{H} is a Banach lattice and all terms on the right hand side of this expression are nonnegative, we must have $||V^n \hat{g}|| \leq ||h||$ for all $n \in \mathbb{N}$.

On the other hand, \hat{g} is a strictly positive element of \mathcal{H}_+ and therefore quasiinterior to \mathcal{H}_+ . Applying theorem 2.1, we have $\|V^n \hat{g}\|^{1/n} \to r(V)$ as $n \to \infty$. Since r(V) > 1, this implies that $\|V^n \hat{g}\| \to \infty$. Contradiction.

Proposition 8.2. *If assumption 3.2 holds and then the valuation operator V is a bounded linear operator on* $L_1(\pi)$ *and, for every strictly positive function* $h \in L_1(\pi)$ *, we have*

$$\lim_{n \to \infty} \left\{ \int V^n h \, \mathrm{d}\pi \right\}^{1/n} = r(V). \tag{42}$$

Proof of proposition 8.2. Recall that an operator $T: L_1(\pi) \to L_1(\pi)$ is called a Hille–Tamarkin operator if *T* takes the form

$$Th(x) = \int k(x, x')h(x')\pi(\mathrm{d}x')$$

for some jointly measurable kernel k on X \times X and, in addition, k satisfies the finite double norm property

$$\int \sup_{x' \in \mathsf{X}} |k(x, x')| \pi(\mathrm{d}x) < \infty.$$
(43)

Hille–Tamarkin operators on $L_1(\pi)$ have the property that T^2 is compact whenever π is σ -finite, as it is in our case. See, for example, theorem 4.5 of Grobler (1970).

Under the conditions of proposition 8.2, the valuation operator V is a Hille–Tamarkin operator. Indeed, V can be expressed as

$$Vh(x) = \int h(x') \int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \pi(x' \mid x) \,\mathrm{d}x'$$
$$= \int h(x') \int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \frac{\pi(x' \mid x)}{\pi(x')} \pi(x') \,\mathrm{d}x'$$

With

$$k(x, x') = \int \phi(x, x', \eta) \nu(\mathrm{d}\eta) \frac{\pi(x' \mid x)}{\pi(x')} \pi(x')$$

and the conditions of proposition 8.2 in force, the integrability condition (43) is satisfied, and V is Hille–Tamarkin as claimed.

As a result, V^2 is a compact linear operator on $L_1(\pi)$. Evidently it is positive. Since h is assumed to be everywhere positive and hence is quasi-interior, it follows from theorem 2.1 that $\{\int V^{2n}h \,d\pi\}^{1/n}$ converges to $r(V^2)$. But $r(V^2) = r(V)^2$, so

$$\left\{\int V^{2n}h\,\mathrm{d}\pi\right\}^{1/(2n)}\to r(V).$$

By our assumptions on V we know that Vh inherits the quasi-interiority of h, so another application of theorem 2.1, this time to V^2 with initial condition Vh, yields

$$\left\{ \int V^{2n} V h \, \mathrm{d}\pi \right\}^{1/n} = \left\{ \int V^{2(n+1)} h \, \mathrm{d}\pi \right\}^{1/n} \to r(V)^2$$
$$\therefore \quad \left\{ \int V^{2(n+1)} h \, \mathrm{d}\pi \right\}^{1/(2n)} \to r(V).$$

Some straightforward analysis then shows that

$$\left\{\int V^{2(n+1)}h\,\mathrm{d}\pi\right\}^{1/(2(n+1))}\to r(V)$$

is also valid. We have now shown that $\{\int V^k h \, d\pi\}^{1/k}$ converges to r(V) along both even and odd subsequences. Hence the sequence itself converges to r(V), and (42) is confirmed.

The second claim in proposition 8.2 is evident from the validity of (42) and the proof of theorem 3.3. \Box

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Proof of proposition 3.4. Let $\mathbb{1}$ be equal to unity everywhere on X. Simple manipulations show that $V^n \mathbb{1}(X_t) = \mathbb{E}_t \prod_{i=1}^n \Phi_{t+i}$. By this equality and the law of iterated expectations,

$$r_{\Phi} = \left\{ \mathbb{E} \prod_{i=1}^{n} \Phi_{t+i} \right\}^{1/n} = \left\{ \mathbb{E} \mathbb{E}_{t} \prod_{i=1}^{n} \Phi_{t+i} \right\}^{1/n} = \left\{ \int V^{n} \mathbb{1} d\pi \right\}^{1/n} \to r(V),$$

as $n \to \infty$, with the convergence due to positivity of 1 and (42).

Proof of proposition 3.5. Let the conditions of the proposition hold but suppose, contrary to the claim in the proposition, that $\{Y_t\}$ is a stationary solution to the price recursion (4) with finite first moment. Iterating on (4) gives

$$Y_t = \mathbb{E}_t \left[\sum_{n=1}^m \prod_{i=1}^n \Phi_{t+i} G_{t+n} + \prod_{i=1}^m \Phi_{t+i} Y_{t+m} \right] \quad \text{for any } m \in \mathbb{N}.$$

Taking expectations and using the law of iterated and the nonnegativity of $\{Y_t\}$, we have

$$\mathbb{E}Y_t \ge \left[\sum_{n=1}^m \mathbb{E}\prod_{i=1}^n \Phi_{t+i}G_{t+n}\right]$$
 for any $m \in \mathbb{N}$.

Together, (16) in proposition 3.4 and the converse component of the Cauchy root criterion imply that this sum diverges. Hence $\mathbb{E}Y_t = \infty$, contradicting our assumption that the solution has finite first moment.

Proof of proposition 5.1. Let the conditions of proposition 8.2 hold. Regarding part (a) of proposition 5.1, that $r_M = r_{\Phi}$ in the present setting follows immediately from $\Phi_{t+1} = M_{t+1}D_{t+1}/D_t$. That $r_{\Phi} = r(V)$ was shown in proposition 3.4.

Regarding part (b), from part (a) we have $r_M < 1$ implies r(V) < 1. Hence we can employ theorem 3.2 and conclude the claim in (b) is true provided that assumption 3.1 is satisfied. That assumption 3.1 is true when $\mathcal{H} = L_1(\pi)$ follows from proposition 8.2, which ensures us that *V* is a bounded linear operator on $L_1(\pi)$.¹⁵

That part (c) holds follows directly from $r_M = r(V)$ and proposition 3.5.

¹⁵An immediate consequence is that *V* maps $L_1(\pi)$ to itself. Moreover, $\hat{g} \in L_1(\pi)$ because $\hat{g} = V \mathbb{1}$ and $\mathbb{1} \in L_1(\pi)$ since π is a probability measure.

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