# Augenblick, Lazarus: Restrictions on Asset-Price Movements Under Rational Expectations: Theory and Evidence

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# OUTLINE

1. Beliefs, risk adjustments and asset prices
    - information in no-arbitrage restrictions
2. Summary of the paper methodology and results
    - martingale restrictions on beliefs
    - localization of state prices
    - implied slopes of the SDF
3. Role of assumptions
    - key assumption: constant SDF slope
4. Questions and suggestions
    - are we testing for rational expectations?

# BACKGROUND

High volatility in prices of risky securities

- high frequency (intraday, daily returns)
- business cycle frequency (monthly, quarterly returns)

Are these asset price fluctuations 'rational'?

- i.e., can they be explained by a plausible model of investors' preferences, market constraints, and investors' beliefs that are correct (in line with the data-generating process (DGP))?

# BACKGROUND

Abstract from market constraints, assume no arbitrage.

- then, for any investor's subjective belief $\widehat{P}$, there exists an SDF $m_{t+1}$ that prices financial assets

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Which is the right model of the SDF, $m_{t+1}$ or $\widehat{m}_{t+1}$?

- that depends on what is plausible $\implies$ impose structure on the SDF
- asset price data alone cannot tell (1) and (2) apart
    - even if we perfectly know the DGP measure $P$

# SDF RESTRICTIONS

Information from no-arbitrage restrictions

$$1 = E_t[\underbrace{m_{t+1}h_{t+1}}_{\widehat{m}_{t+1}}R_{t+1}]$$

- Time series information identifies the DGP measure $P$.
- Data on returns $R_{t+1}$ together with no-arbitrage restrictions inform $\widehat{m}_{t+1}$.
- We must impose additional restrictions to separately infer something about $m_{t+1}$ and $h_{t+1}$.

# SDF RESTRICTIONS

This paper

$$1 = E_t[\underbrace{m_{t+1}h_{t+1}}_{\widehat{m}_{t+1}}R_{t+1}]$$

- imposes the zero of correct beliefs $\implies h_{t+1} \equiv 1$
- restricts the class of SDFs (conditional transition independence (CTI)) $\implies$ restriction on $m_{t+1}$
- uses martingale restrictions under DGP (time series information) to deduce whether $\exists m_{t+1}$ in the given class that also prices returns

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Outcomes

- if $\exists$ such $m_{t+1}$, characterize the minimum needed local risk aversion consistent with $h_{t=1} \equiv 1$
- if $\nexists$ such $m_{t+1}$, conclude that $h_{t+1} \neq 1$.

Let $X_t$ be a square-integrable martingale. Then

$$E_t\left[\sum_{j=t}^{T-1}(X_{j+1}-X_j)^2\right]=E_t\left[X_T^2-X_t^2\right]$$

- restriction on quadratic variation of a martingale

# MARTINGALE RESTRICTIONS

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'Beliefs are martingales' (under their own measure)

- apply the above result to a two-state outcome $\implies$ localization
- $\pi_t$ — probability of the low state realization at $T$

$$E_t\left[m_T\right] \doteq E_t\left[\sum_{j=t}^{T-1}(\pi_{j+1}-\pi_j)^2\right] = E_t\left[(1-\pi_t)\,\pi_t - (1-\pi_T)\,\pi_T\right] \doteq E_t\left[r_T\right]$$

- if we could measure beliefs, this would constitute a test of rational expectations (Augenblick and Rabin (2018))
- restriction on how much beliefs can vary over time

# ARROW–DEBREU PRICES

Instead, we observe risk-neutral (Arrow–Debreu) prices $\pi_t^*$. Then

$$E_t\left[m_T^*\right] \doteq E_t\left[\sum_{j=t}^{T-1}\left(\pi_{j+1}^* - \pi_j^*\right)^2\right] \neq E_t\left[\left(1-\pi_t^*\right)\pi_t^* - \left(1-\pi_T^*\right)\pi_T^*\right] \doteq E_t\left[r_T^*\right]$$

Relationship between $\pi_t^*$ and $\pi_t$

$$\pi_t^* \propto E_t\left[M_T/M_t \mid s_T = \text{low}\right]\pi_t$$

- reflects risk adjustment and normalization by the risk-free rate

Rewrite relation between $\pi_t^*$ and $\pi_t$ in terms of odds ratio

$$\underbrace{\frac{\pi_t^*}{1-\pi_t^*}}_{\substack{\text{risk-neutral}\\\text{prices}}} = \underbrace{\frac{E_t\left[M_T/M_t \mid s_T = \text{low}\right]}{E_t\left[M_T/M_t \mid s_T = \text{high}\right]}}_{\text{SDF slope } \phi} \underbrace{\frac{\pi_t}{1-\pi_t}}_{\text{beliefs}}$$

If we observe (a lot of) variation in $\pi_t^*$ over time, what can we infer about variation in $\pi_t$?

- we must restrict movements in the SDF slope $\phi$ over time
- $\phi$ can differ across terminal state realizations

# RESTRICTING THE SDF SLOPE

Assumption: Conditional transition independence (CTI)

- $\phi$ is constant over time (it can also follow a martingale)

$$\underbrace{\frac{\pi_t^*}{1-\pi_t^*}}_{\substack{\text{risk-neutral}\\\text{prices}}} = \underbrace{\frac{E_t\left[M_T/M_t \mid s_T = \text{low}\right]}{E_t\left[M_T/M_t \mid s_T = \text{high}\right]}}_{\text{SDF slope } \phi} \underbrace{\frac{\pi_t}{1-\pi_t}}_{\text{beliefs}}$$

Idea: A high $\phi$ translates large variation in $\pi_t^*$ into modest variation in $\pi_t$, which can then satisfy the martingale restriction on correct beliefs.

$$E_t\left[m_T^* - r_t^*\right] \leq \left(\pi_t^*\right)^2 \left(1 - \frac{1}{\pi_t^* + \phi\left(1-\pi_t^*\right)}\right)$$

Define states: Terminal values of a market index $R_T^m = s_T$ at some date $T$.

Extract Arrow–Debreu prices (Breeden, Litzenberger (1978))

Construct sample averages for $E_t\left[m_T^* - r_t^*\right]$

- daily, weekly, monthly frequencies
- conditional on terminal returns for various adjacent payoff states $R_T^m \in \{s_j, s_{j+1}\} \implies$ localization of the return distribution

Infer the required $\phi$'s to satisfy inequality

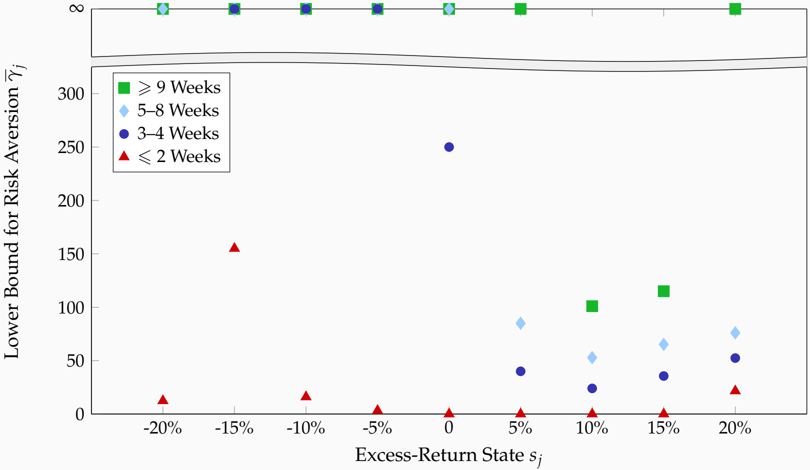
$$E_t\left[m_T^* - r_t^*\right] \leq (\pi_t^*)^2 \left(1 - \frac{1}{\pi_t^* + \phi\,(1 - \pi_t^*)}\right)$$

# SUMMARY OF THE RESULTS

Required SDF slopes $\phi$ need to be very high to rationalize correct beliefs.

- Consequence of high volatility in $\pi_t^*$.

**Figure 5: Estimates of Relative Risk Aversion: Splits by Time to Expiration**



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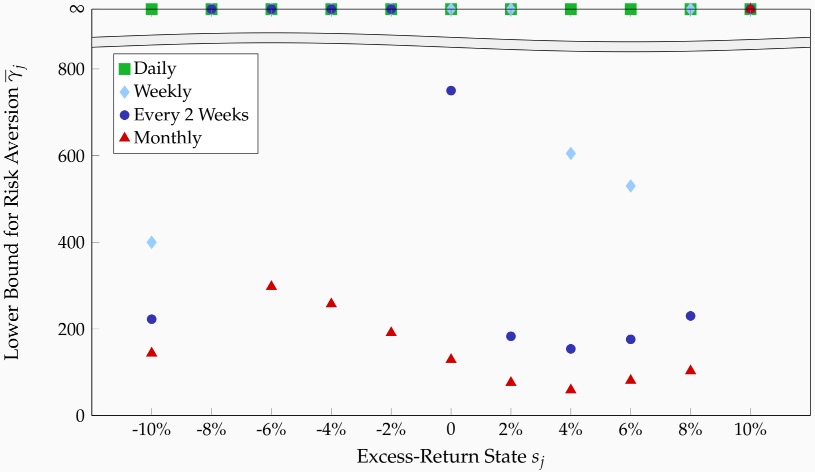
Higher $\phi$ needed to rationalize higher frequency (daily) variation.

- Is there more 'unexplained' variation at higher frequencies? Unclear, only a necessary bound.

**Figure 6: Estimates of Relative Risk Aversion: Splits by Sampling Frequency**



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High risk aversion needed across all terminal payoff states.

- Informative about the type of risks faced by investors (e.g., tail risk).

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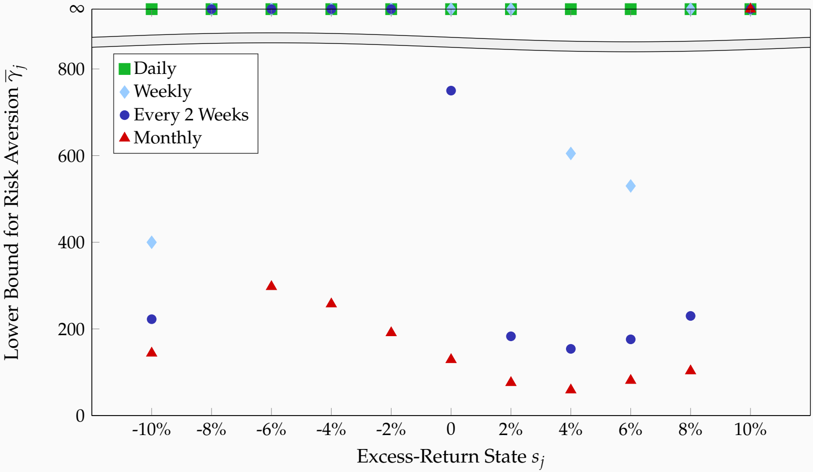

# DISCUSSION OF ASSUMPTIONS

The whole identification hinges on the choice of class of 'plausible' SDFs.

$$\frac{\pi_t^*}{1-\pi_t^*} = \underbrace{\frac{E_t\left[M_T/M_t \mid R_T^m = \text{low}\right]}{E_t\left[M_T/M_t \mid R_T^m = \text{high}\right]}}_{\substack{\text{constant SDF slope } \phi \\ \phi = \frac{u'\left(R_T^m = \text{low}\right)}{u'\left(R_T^m = \text{high}\right)}}} \frac{\pi_t}{1-\pi_t}$$

To first-order approximation, this corresponds to constant local relative risk aversion $\gamma\left(\bar{R}\right)$

- market risk premium

$$E_t\left[R_T^e\right] = \gamma\left(\bar{R}\right) Var_t\left[R_T^e\right]$$

- $\gamma\left(\bar{R}\right)$ is the local curvature of the utility/value function as a function of the return realization, by assumption independent of $t$
- time-variation in market risk premium can only be driven by $Var_t\left[R_T^e\right]$

## WHEN DOES THE ASSUMPTION HOLD?

Separable utility over returns at terminal state

- caveat: if utility is over consumption, then time variation in the mapping between returns and consumption can lead to violations

Epstein–Zin in iid growth environment

- not very interesting, isomorphic to recalibrated separable CRRA utility

Disaster risk model (Gabaix (2012)) with CRRA utility

- holds for (high) return realizations conditional on which disasters between today and maturity have a negligible chance
- interesting: utilizes localization to return dynamics of particular AD securities
- for a rejection of rationality, it requires establishing what negligible chance means in the model and data

In general, $\phi$ cannot move systematically with $\pi_t^*$.

# DEVIATIONS FROM CONSTANT RISK AVERSION

$$\frac{\pi_t^*}{1-\pi_t^*} = \underbrace{\frac{E_t\left[M_T/M_t \mid R_T^m = \text{low}\right]}{E_t\left[M_T/M_t \mid R_T^m = \text{high}\right]}}_{\text{constant SDF slope } \phi} \frac{\pi_t}{1-\pi_t}$$

Inference with constant $\phi$

- bad times $\implies$ observed $\uparrow \pi_t^*$ $\implies$ infer required increase in $\pi_t$, given $\phi$

Systematic variation in $\phi$

- bad times $\implies$ observed $\uparrow \pi_t^*$ $\implies$ associated increase in $\phi$ $\implies$ lower required increase in $\pi_t$
- lower average $\phi$ needed to dampen implied variation in $\pi_t$ sufficiently to satisfy the martingale restriction

# WHAT IS OFF THE TABLE?

Nonseparable preferences (Epstein–Zin) with interesting state dynamics

Explicit time-variation in risk aversion

Models with agent heterogeneity (e.g., in risk aversion)

- implied risk aversion a function of wealth distribution
- how to interpret results in models with heterogeneous beliefs?

Models with financial constraints

- implied risk aversion changes with tightness of constraints

Habit formation models (Campbell and Cochrane (1999))

- implied risk aversion function of habit level
- authors show that in CC (1999), the bound still holds approximately
  - but SDF slopes implied by the model are trivial for non-extreme states

$$\frac{\pi_t^*}{1-\pi_t^*} = \underbrace{\frac{E_t\left[M_T/M_t \mid R_T^m = \text{low}\right]}{E_t\left[M_T/M_t \mid R_T^m = \text{high}\right]}}_{\text{time-varying SDF slope } \phi_t} \frac{\pi_t}{1-\pi_t}$$

Question: How much variation in $\phi_t$ is needed such that implied $\pi_t$ satisfy the martingale restriction?

- a test of a larger class of (more interesting) models

Existing models may well still fail

- many asset pricing models designed to match unconditional risk premia and the cross-section
- appropriate time-variation harder to achieve

# OTHER CONSIDERATIONS

Implementation relies purely on option prices

- studies option price dynamics (with all their specifics) relative to the index
- what about correct beliefs in the pricing of the index itself?

Authors relate identified excess belief movement to macro statistics

- why not survey data directly?

Why bounds at all?

- Given the information obtained from localization, one could directly attempt to estimate the SDF nonparametrically (Christensen (2017))

Approximation details (discretization, etc.)

- authors take these seriously, but details are subtle

Localization

- we do not need to test adjacent states only, additional information in alternative state combinations

# CONCLUSION I.

Given stated assumptions, the paper is cleanly executed, with a lot of detail.

- robustness checks, careful econometrics
- models could be studied in more detail

Interesting results

- much higher high frequency (daily) variation than variation at 'macro' frequencies
  - Market microstructure effects in daily data? Different models needed?
  - Caution regarding bid-ask bounce? (not covered here)
- localization of return dynamics can discard some models
  - high AD price variation for high-return states speaks against disaster-type models that focus risk adjustments to adverse states

A test of rational expectations?

- restricts attention to a class of models that is too narrow
- reverting the problem and studying required variation in implied risk aversion may be more fruitful

Even if we impose stringent restrictions on SDF and reject correct beliefs, the puzzle still stands.

- why do beliefs fluctuate so much?
- need to devise interesting models of belief dynamics to make progress …
- … just like interesting models of risk adjustments that go beyond imposed restrictions …
- … and in some cases, the two cannot be distinguished using asset price data anyway
  - e.g. Epstein–Zin with unitary IES vs. Hansen–Sargent robust preference vs. ex-post Bayesian worst-case belief

# SIDE NOTE: RELATIONSHIP TO ROSS (2015)

Ross (2015) 'Recovery Theorem'

- uses only cross-sectional price information, no time series
- imposes separable marginal utility (transition independence (TI))

$$m_{t+1} = \beta \frac{u'(s_{t+1})}{u'(s_t)}$$

  and stationarity of the state dynamics
- under TI, it recovers subjective belief $\widehat{P}$ (possibly different from $P$)
- if TI does not hold, it recovers a long-run risk neutral measure (different from $\widehat{P}$ or $P$)

This paper

- imposes a restriction based on time-series information in $P$ (time-variation in prices)
- asks whether imposing $P = \widehat{P}$ can be consistent with SDFs that satisfy CTI
- allows for nonstationarity of the state and hence for some martingale components in SDF
- both papers assume time-invariant local curvature over terminal states